We can compute \( x := n! \) (factorial) as follows.

\[
x := n! \iff \begin{cases} 
\text{if } n = 0 & \text{then } x := 1 \\
\text{else } n := n - 1 \quad x := n! 
\end{cases}
\]

Each call \( x := n! \) pushes a return address onto a stack, and each return pops an address from the stack. Add a space variable \( s \) and a maximum space variable \( m \), with appropriate assignments to them in the program. Find and prove an upper bound on the maximum space used.

\[
\text{Proof: by cases. First case:}
\]

\[
(n = 0 \land (x := 1) \Rightarrow (m \geq s \Rightarrow m' = \max m (s+n)) \quad \text{portation and expand}
\]

\[
(n = 0 \land x' \leq 1 \land n' = n \land s' = s \land m' = m \land m \geq s \Rightarrow m' = \max m (s+n)) \quad \text{context}
\]

\[
(n = 0 \land x' \leq 1 \land n' = n \land s' = s \land m' = m \land m \geq s \Rightarrow \top) \quad \text{base}
\]

\[
\Rightarrow \top
\]

Second case:

\[
\begin{align*}
m \geq s \Rightarrow m' &= \max m (s+n) \\
\iff (n + 0 \land (n := n - 1 \quad s := s + 1 \quad m := \max m s \quad m \geq s \Rightarrow m' = \max m (s+n) \quad s := s - 1 \quad n := n + 1 \quad x := x \times n)) \quad \text{portation}
\end{align*}
\]

\[
\Rightarrow m' = \max m (s+n) \quad \text{three substitutions; expand final assignment and two more substitutions}
\]

\[
\begin{align*}
n + 0 \land m \geq s \\
\land (m' = \max m (s+n) \quad x' = x \times (n+1) \land n' = n + 1 \land s' = s - 1 \land m' = m) \\
\Rightarrow m' = \max m (s+n) \quad \text{simplify first } \max \text{ to } \top \text{ and } +1-1 \text{ to } 0 .
\end{align*}
\]

In the context where natural \( n \) is not 0, \( s+n \geq s+1 \) so remove \( s+1 \) from \( \max \).

\[
\begin{align*}
n + 0 \land m \geq s \\
\land (m' = \max m (s+n) \quad x' = x \times (n+1) \land n' = n + 1 \land s' = s - 1 \land m' = m) \\
\Rightarrow m' = \max m (s+n) \quad \text{Eliminate} . \text{ and then use one-point.}
\end{align*}
\]

\[
\begin{align*}
n + 0 \land m \geq s \land m' = \max m (s+n) \\
\Rightarrow m' = \max m (s+n) \quad \text{specialize}
\end{align*}
\]

\[
\Rightarrow \top
\]