28 (doorway to heaven) There is a door that leads either to heaven or to hell. There is a guard who knows where the door leads. If you ask the guard a question, the guard may tell the truth, or the guard may lie. Ask the guard one question to determine whether the door leads to heaven or hell.

After trying the question, scroll down to the solution.

§ Let binary h mean that the door leads to heaven (h may be $\top$ or $\bot$). Let binary q be our question (this is what we are trying to construct). Let binary a be the guard's answer (a may be $\top$ or $\bot$). Let binary g mean that the guard tells the truth (g may be $\top$ or $\bot$). We will ask the guard question q . We want the guard's answer a to tell us whether the door leads to heaven h . So we want

$$a=h$$

If the guard tells the truth, then the answer has the same value as the question.

$$g \Rightarrow a=q$$

If the guard lies, then the answer has the opposite value from the question.

$$\neg g \Rightarrow a \neq q$$

Putting it all together,

$$\begin{array}{lll}
 & a=h \wedge (g \Rightarrow a=q) \wedge (\underline{\neg g \Rightarrow a \neq q}) & \text{exclusion} \\
= & a=h \wedge (g \Rightarrow a=q) \wedge (g \Leftarrow a=q) & \text{antisymmetry} \\
= & a=h \wedge g=(a=q) & \text{context} \\
= & a=h \wedge g=(h=q) & \text{associativity and symmetry of } = \\
= & a=h \wedge q=(h=g) &
\end{array}$$

So the question q we ask is $h=g$: “Does the door lead to heaven if and only if you are telling the truth?”, or “Do the questions “Does the door lead to heaven?” and “Are you telling the truth?” have the same answer?”.

Suppose the door leads to heaven (h), and the guard tells the truth (g); then the guard truly answers $h=g$ which is $\top=\top$, which is $\top$, which tells us correctly that the door leads to heaven. Suppose the door leads to heaven (h), and the guard lies ($\neg g$); then the guard falsely answers $\neg(h=g)$ which is $\neg(\top=\bot)$, which is $\top$, which tells us correctly that the door leads to heaven. Suppose the door leads to hell ($\neg h$), and the guard tells the truth (g); then the guard truly answers $h=g$ which is $\bot=\top$, which is $\bot$, which tells us correctly that the door leads to hell. Suppose the door leads to hell ($\neg h$), and the guard lies ($\neg g$); then the guard falsely answers $\neg(h=g)$ which is $\neg(\bot=\bot)$, which is $\bot$, which tells us correctly that the door leads to hell.

The guard may tell the truth, or the guard may lie. The guard can even decide whether to tell the truth or to lie after hearing the question. So what stops the guard from misinforming us about where the door leads? If the door leads to heaven, the guard is physically capable of saying “no”, and if the door leads to hell, the guard is physically capable of saying “yes”. But the guard is stopped from doing so by the given information: the guard may tell the truth, or the guard may lie, but the guard will not utter an inconsistency. Suppose the door leads to heaven (h). For the guard to misinform us, the guard has to answer $\bot$. The question is $h=g$, and we are supposing h , so the question is $\top=g$, which is the same as g , and the guard answers $\bot$. So the guard is saying “I am lying”, which is the Liar's Paradox (see Exercise 3(ii)); it is a self-contradictory, or inconsistent answer. Suppose the door leads to hell ($\neg h$). For the guard to misinform us, the guard has to answer $\top$. The question is $h=g$, and we are supposing $\neg h$, so the question is $\bot=g$, which is the same as $\neg g$, and the guard answers $\top$. So again the guard is saying “I am lying”, which is an inconsistent answer. In order to not be an inconsistency, the guard's answer must tell us whether the door leads to heaven or hell.