Given three sorted lists having at least one item common to all three, write a program to find the smallest item occurring in all three lists.

Let the three lists be $A$, $B$, and $C$ and let $a$, $b$, and $c$ be natural variables to act as indexes of the three lists. The specification is $R$, defined as

$$R = A(0..a') \land B(0..b') \land C(0..c') = \text{null}$$

$$\land A \cdot a' = B \cdot b' = C \cdot c'$$

$$\land t' = t + a' + b' + c'$$

which says there isn't a common item before indexes $a'$, $b'$, and $c'$, and there is one at those indexes, and the sum of those indexes is the execution time. We also need the specification $Q$, defined as

$$Q = A(a..a') \land B(b..b') \land C(c..c') = \text{null}$$

$$\land A \cdot a' = B \cdot b' = C \cdot c'$$

$$\land t' = t + a' - a + b' + b + c' - c$$

which says the same thing, but starting at indexes $a$, $b$, and $c$. The refinements are

$$R \iff a := 0. \quad b := 0. \quad c := 0. \quad Q$$

$$Q \iff \begin{cases} 
\text{if } A < B \text{ then } a := a + 1. \quad t := t + 1. \quad Q \\
\text{else if } B < C \text{ then } b := b + 1. \quad t := t + 1. \quad Q \\
\text{else if } C < A \text{ then } c := c + 1. \quad t := t + 1. \quad Q \\
\end{cases}$$

The proof of the first refinement is immediate after using the Substitution Law 3 times. The proof of the last refinement breaks into 12 pieces (3 conjuncts $\times$ 4 cases). Let's start with the first conjunct of $Q$ and the first case of the if.

$$A < B \land (a := a + 1. \quad t := t + 1. \quad (a..a') \land (b..b') \land (c..c') = \text{null})$$

substitution law twice

$$= A < B \land (a + 1..a') \land (b..b') \land (c..c') = \text{null}$$

Because $B$ is sorted, $B$ is the minimum item of $B(b..b')$.

And since $A < B$, therefore $A a$ is unequal to any item in $B(b..b')$.

$$\Rightarrow (a..a') \land (b..b') \land (c..c') = \text{null}$$

Now we prove the middle conjunct of $Q$ with the same first case of the if.

$$A < B \land (a := a + 1. \quad t := t + 1. \quad A a' = B b' = C c')$$

substitution law twice

$$= A < B \land A a' = B b' = C c'$$

specialize

$$\Rightarrow A a' = B b' = C c'$$

Now we prove the last conjunct of $Q$ with the same first case of the if.

$$A < B \land (a := a + 1. \quad t := t + 1. \quad t' = t + a' - a + b' + b + c' - c)$$

substitution law twice

$$= A < B \land t' = t + a' - a + b' + b + c' - c$$

simplify and specialize

$$\Rightarrow t' = t + a' - a + b' + b + c' - c$$

The proofs of the next two cases are exactly the same. That leaves the last case.

$$A a \geq B b \land B b \geq C c \land C c \geq A a \land \text{ok}$$

antisymmetry, and expand $\text{ok}$

$$A a = B b = C c \land a' = a \land b' = b \land c' = c \land t' = t$$

$$\Rightarrow Q$$