Given two sorted lists having at least one item in common, write a program to find the smallest item occurring in both lists.

After trying the question, scroll down to the solution.
Let the two lists be $A$ and $B$, and let $a$ and $b$ be natural variables to act as indexes of the two lists. The specification is $R$, defined as

$$R = A(0..a') \cdot B(0..b') = \text{null}$$

- $A a' = B b$
- $t' = t + a' + b'$

which says there isn't a common item before indexes $a'$ and $b'$, and there is one at those indexes, and the sum of those indexes is the execution time. We also need the specification $Q$, defined as

$$Q = A(a..a') \cdot B(b..b') = \text{null}$$

- $A a' = B b'$
- $t' = t + a' - a + b' - b$

which says the same thing, but starting at indexes $a$ and $b$. The refinements are

$R \leftarrow a := 0. b := 0. Q$

$Q \leftarrow$

- if $A a < B b$ then $a := a + 1. t := t + 1. Q$
- else if $B b < C c$ then $b := b + 1. t := t + 1. Q$
- else ok fi

The proof of the first refinement is immediate after using the Substitution Law 2 times. The proof of the last refinement breaks into 9 pieces (3 conjuncts $\times$ 3 cases). Let's start with the first conjunct of $Q$ and the first case of the if.

$$A a < B b \land (a := a + 1. t := t + 1. A(a..a') \cdot B(b..b') = \text{null}) \text{ substitution law twice}$$

Because $B$ is sorted, $B b$ is the minimum item of $B(b..b')$.

And since $A a < B b$, therefore $A a$ is unequal to any item in $B(b..b')$.

$$\Rightarrow A(a..a') \cdot B(b..b') = \text{null}$$

Now we prove the middle conjunct of $Q$ with the same first case of the if.

$$A a < B b \land (a := a + 1. t := t + 1. A a' = B b') \text{ substitution law twice}$$

$$\Rightarrow A a' = B b'$$

Now we prove the last conjunct of $Q$ with the same first case of the if.

$$A a < B b \land (a := a + 1. t := t + 1. t' = t + a' - a + b' - b) \text{ substitution law twice}$$

$$\Rightarrow t' = t + a' - a + b' - b$$

The proof of the middle case is exactly the same. That leaves the last case.

$$A a \geq B b \land B b \geq A a \land \text{ok}$$

$$A a = B b \land a' = a \land b' = b \land t' = t$$

$$\Rightarrow Q$$