272 (common items) Let A be a sorted list of different integers. Let B be another such list. Write a program to find the number of integers that occur in both lists.

After trying the question, scroll down to the solution.

§ Let a and b be natural variables serving as indexes of A and B respectively. Define

$$Q \;=\; n' = n + \text{¢}(A(a,..\#A) \text{ ‘ } B(b,..\#B))$$

The refinements are

$$n' = \text{¢}(A(0,..\#A) \text{ ‘ } B(0,..\#B)) \;\Leftarrow\; a:= 0.\; b:= 0.\; n:= 0.\; Q$$

$$\begin{aligned} Q \;\Leftarrow\; & \textbf{if } a{=}\#A \vee b{=}\#B \textbf{ then } ok \\ & \textbf{else if } A\,a < B\,b \textbf{ then } a:= a{+}1.\; Q \\ & \quad \textbf{else if } A\,a > B\,b \textbf{ then } b:= b{+}1.\; Q \\ & \quad\quad \textbf{else } n:= n{+}1.\; a:= a{+}1.\; b:= b{+}1.\; Q \textbf{ fi fi fi} \end{aligned}$$

The first refinement is proven by 3 substitutions. The next refinement breaks into 4 cases. First case:

$$Q \;\Leftarrow\; (a{=}\#A \vee b{=}\#B) \wedge ok \qquad \text{expand } Q \text{ and } ok$$

$$= \quad n' = n + \text{¢}(A(a,..\#A) \text{ ‘ } B(b,..\#B) \;\Leftarrow\; (a{=}\#A \vee b{=}\#B) \wedge a'{=}a \wedge b'{=}b \wedge n'{=}n$$

Since $a{=}\#A \vee b{=}\#B$, therefore $A(a,..\#A) \text{ ‘ } B(b,..\#B) = null$

$$= \quad \top$$

Second case:

$$Q \;\Leftarrow\; a{\ne}\#A \wedge b{\ne}\#B \wedge A\,a < B\,b \wedge (a:= a{+}1.\; Q) \qquad \text{expand and substitute}$$

$$\begin{aligned} = \quad & n' \;=\; n + \text{¢}(A(a,..\#A) \text{ ‘ } B(b,..\#B)) \\ \Leftarrow \quad & a{\ne}\#A \;\wedge\; b{\ne}\#B \;\wedge\; Aa{<}Bb \;\wedge\; n' = n + \text{¢}(A(a{+}1,..\#A) \text{ ‘ } B(b,..\#B)) \end{aligned}$$

According to the antecedent, $A\,a < B\,b$, and because B is sorted, $A\,a <$ each item in $B(b,..\#B)$. Hence $A(a{+}1,..\#A) \text{ ‘ } B(b,..\#B) \;=\; A(a,..\#A) \text{ ‘ } B(b,..\#B)$.

$$= \quad \top$$

Next case:

$$Q \;\Leftarrow\; a{\ne}\#A \wedge b{\ne}\#B \wedge A\,a > B\,b \wedge (b:= b{+}1.\; Q) \qquad \text{like previous case}$$

$$= \quad \top$$

Last case:

$$Q \;\Leftarrow\; a{\ne}\#A \wedge b{\ne}\#B \wedge A\,a = B\,b \wedge (a:= a{+}1.\; b:= b{+}1.\; Q)) \qquad \text{expand and subst}$$

$$\begin{aligned} = \quad & n' \;=\; n + \text{¢}(A(a,..\#A) \text{ ‘ } B(b,..\#B)) \\ \Leftarrow \quad & a{\ne}\#A \;\wedge\; b{\ne}\#B \;\wedge\; A\,a = B\,b \;\wedge\; n' = n + \text{¢}(A(a{+}1,..\#A) \text{ ‘ } B(b{+}1,..\#B)) \end{aligned}$$

Since $A\,a = B\,b$, and the lists do not have duplicates, therefore $\text{¢}(A(a,..\#A) \text{ ‘ } B(b,..\#B)) \;=\; 1 + \text{¢}(A(a{+}1,..\#A) \text{ ‘ } B(b{+}1,..\#B))$

$$= \quad \top$$

The timing is $t' \le t + \#A + \#B$, and we replace Q with

$$a{\le}\#A \wedge b{\le}\#B \;\Rightarrow\; t' \le t + \#A - a + \#B - b$$

and put $t:= t{+}1$ before each of the 3 recursive calls. The first refinement

$$\begin{aligned} & t' \le t + \#A + \#B \\ \Leftarrow \quad & a:= 0.\; b:= 0.\; n:= 0.\; a{\le}\#A \wedge b{\le}\#B \;\Rightarrow\; t' \le t + \#A - a + \#B - b \end{aligned}$$

is proven by 2 substitutions. The next refinement breaks into 4 cases. First case:

$$\begin{aligned} & a{\le}\#A \wedge b{\le}\#B \;\Rightarrow\; t' \le t + \#A - a + \#B - b \\ \Leftarrow \quad & (a{=}\#A \vee b{=}\#B) \wedge a'{=}a \wedge b'{=}b \wedge n'{=}n \wedge t'{=}t \qquad \text{context, then delete antecedent} \\ \Leftarrow \quad & a{\le}\#A \wedge b{\le}\#B \;\Rightarrow\; t \le t + \#A - a + \#B - b \\ = \quad & \top \end{aligned}$$

Second case:

$$\begin{aligned} & a{\le}\#A \wedge b{\le}\#B \;\Rightarrow\; t' \le t + \#A - a + \#B - b \\ \Leftarrow \quad & a{\ne}\#A \wedge b{\ne}\#B \wedge A\,a < B\,b \\ & \wedge (a:= a{+}1.\; t:= t{+}1.\; a{\le}\#A \wedge b{\le}\#B \;\Rightarrow\; t' \le t + \#A - a + \#B - b) \\ & \qquad \text{portation and 2 substitutions} \\ = \quad & a{\le}\#A \wedge b{\le}\#B \wedge a{\ne}\#A \wedge b{\ne}\#B \wedge A\,a < B\,b \\ & \wedge (a{+}1{\le}\#A \wedge b{\le}\#B \;\Rightarrow\; t' \le t{+}1{+}\#A{-}(a{+}1){+}\#B{-}b)) \\ \Rightarrow \quad & t' \le t + \#A - a + \#B - b \qquad \text{simplify} \\ = \quad & a{<}\#A \wedge b{<}\#B \wedge A\,a < B\,b \\ & \wedge (a{+}1{\le}\#A \wedge b{\le}\#B \;\Rightarrow\; t' \le t{+}\#A{-}a{+}\#B{-}b)) \end{aligned}$$

$$\Rightarrow \quad t' \le t + \#A - a + \#B - b \qquad \text{discharge}$$

$$= \quad a<\#A \wedge b<\#B \wedge A\,a < B\,b \wedge\ t' \le t+\#A-a+\#B-b)$$

$$\Rightarrow \quad t' \le t + \#A - a + \#B - b \qquad \text{specialize}$$

$$= \quad \top$$

Next case:

$$a\le\#A \wedge b\le\#B \ \Rightarrow\ t' \le t + \#A - a + \#B - b$$

$$\Leftarrow \quad a\ne\#A \ \wedge\ b\ne\#B \ \wedge\ A\,a > B\,b$$

$$\wedge\ (b{:=}\ b{+}1.\ \ t{:=}\ t{+}1.\ \ a\le\#A \wedge b\le\#B \ \Rightarrow\ t' \le t + \#A - a + \#B - b)$$

like previous case

$$= \quad \top$$

Last case:

$$a\le\#A \wedge b\le\#B \ \Rightarrow\ t' \le t + \#A - a + \#B - b$$

$$\Leftarrow \quad a\ne\#A \ \wedge\ b\ne\#B \ \wedge\ A\,a = B\,b$$

$$\wedge\ (a{:=}\ a{+}1.\ \ b{:=}\ b{+}1.\ \ t{:=}\ t{+}1.\ \ a\le\#A \wedge b\le\#B \ \Rightarrow\ t' \le t + \#A - a + \#B - b)$$

portation, 3 substitutions, simplify

$$= \quad a<\#A \ \wedge\ b<\#B \ \wedge\ A\,a = B\,b$$

$$\wedge\ (a{+}1\le\#A \wedge b{+}1\le\#B \ \Rightarrow\ t' \le t{+}1 + \#A - (a{+}1) + \#B - (b{+}1))$$

$$\Rightarrow \quad t' \le t + \#A - a + \#B - b \qquad \text{simplify and discharge}$$

$$= \quad a<\#A \ \wedge\ b<\#B \wedge\ A\,a = B\,b \ \wedge\ t' \le t + \#A - a + \#B - b - 1$$

$$\Rightarrow \quad t' \le t + \#A - a + \#B - b \qquad \text{connection}$$

$$= \quad \top$$