Given three sorted lists having at least one item common to all three, write a program to find the smallest item occurring in all three lists.

Let the three lists be \( A \), \( B \), and \( C \) and let \( a \), \( b \), and \( c \) be natural variables to act as indexes of the three lists. The specification is \( R \), defined as

\[
R = A(0..a') \land B(0..b') \land C(0..c') = \text{null} \\
\land \quad Aa' = Bb' = Cc' \\
\land \quad t' = t + a' + b' + c'
\]

which says there isn't a common item before indexes \( a' \), \( b' \), and \( c' \), and there is one at those indexes, and the sum of those indexes is the execution time. We also need the specification \( Q \), defined as

\[
Q = A(a..a') \land B(b..b') \land C(c..c') = \text{null} \\
\land \quad Aa' = Bb' = Cc' \\
\land \quad t' = t + a' - a + b' - b + c' - c
\]

which says the same thing, but starting at indexes \( a \), \( b \), and \( c \). The refinements are

\[
R \iff a : = 0. \quad b : = 0. \quad c : = 0. \quad Q \\
Q \iff \text{if } Aa < Bb \text{ then } a : = a + 1. \quad t : = t + 1. \quad Q \\
\quad \text{else if } Bb < Cc \text{ then } b : = b + 1. \quad t : = t + 1. \quad Q \\
\quad \quad \text{else if } Cc < Aa \text{ then } c : = c + 1. \quad t : = t + 1. \quad Q \\
\quad \quad \quad \text{else } \text{ok fi fi fi}
\]

The proof of the first refinement is immediate after using the Substitution Law 3 times. The proof of the last refinement breaks into 12 pieces (3 conjuncts \( \times 4 \) cases). Let's start with the first conjunct of \( Q \) and the first case of the \( \text{if} \).

\[
Aa < Bb \land (a : = a + 1. \quad t : = t + 1. \quad A(a..a') \land B(b..b') \land C(c..c') = \text{null})
\]

substitution law twice

And since \( Aa < Bb \), therefore \( Aa \) is unequal to any item in \( B(b..b') \).

\[
\Rightarrow A(a..a') \land B(b..b') \land C(c..c') = \text{null}
\]

Now we prove the middle conjunct of \( Q \) with the same first case of the \( \text{if} \).

\[
Aa < Bb \land (a : = a + 1. \quad t : = t + 1. \quad Aa' = Bb' = Cc')
\]

substitution law twice

specialize

\[
\Rightarrow Aa' = Bb' = Cc'
\]

Now we prove the last conjunct of \( Q \) with the same first case of the \( \text{if} \).

\[
Aa < Bb \land (a : = a + 1. \quad t : = t + 1. \quad t' = t + a' - a + b' - b + c' - c)
\]

substitution law twice

\[
\Rightarrow t' = t + a' - a + b' - b + c' - c
\]

The proofs of the next two cases are exactly the same. That leaves the last case.

\[
Aa \geq Bb \land Bb \geq Cc \land Cc \geq Aa \land \text{ok} \\
Aa = Bb = Cc \land a' = a \land b' = b \land c' = c \land t' = t
\]

\[
\Rightarrow Q
\]