There are three inhabitants of an island, named P, Q, and R. Each is either a knight or a knave. Knights always tell the truth. Knaves always lie. For each of the following, write the given information formally, and then answer the questions, with proof.

Let $p$ mean “P is a knight”, let $q$ mean “Q is a knight”, and let $r$ mean “R is a knight”.

(a) P says: “If I am a knight, I'll eat my hat.”. Does P eat his hat?

The given information is $p = (p \Rightarrow e)$, which we take as an axiom.

$p = (p \Rightarrow e)$ double implication
$\equiv (p \Rightarrow (p \Rightarrow e)) \land ((p \Rightarrow e) \Rightarrow p)$ portation and idempotence
$\equiv (p \Rightarrow e) \land ((p \Rightarrow e) \Rightarrow p)$ discharge
$\equiv (p \Rightarrow e) \land p$ symmetry and discharge
$\equiv p \land e$ So P is a knight and he eats his hat.

(b) P says: “If Q is a knight then I am a knave.”. What are P and Q?

The given information is $p = (q \Rightarrow \neg p)$, which we take as an axiom.

$p = (q \Rightarrow \neg p)$ contrapositive
$\equiv p = (p \Rightarrow \neg q)$ double implication
$\equiv (p \Rightarrow (p \Rightarrow \neg q)) \land ((p \Rightarrow \neg q) \Rightarrow p)$ portation and idempotence
$\equiv (p \Rightarrow \neg q) \land ((p \Rightarrow \neg q) \Rightarrow p)$ discharge
$\equiv (p \Rightarrow \neg q) \land p$ symmetry and discharge
$\equiv p \land \neg q$ So P is a knight and Q is a knave.

(c) P says: “There is gold on this island if and only if I am a knight.”. Can it be determined whether P is a knight or a knave? Can it be determined whether there is gold on the island?

Let $g$ mean “there is gold on this island”. Starting with the axiom,

$p = (g \equiv p)$ use symmetry of $=$
$\equiv p = (p = g)$ use associativity of $=$
$\equiv (p = p) = g$ use reflexivity of $=$
$\equiv \top = g$ $\top$ is identity of $=$
$\equiv g$

So there is gold on the island but we don't know what P is.

(d) P, Q, and R are standing together. You ask P: “Are you a knight or a knave?” P mumbles his reply, and you don't hear it. So you ask Q: “What did P say?” Q replies: “P said that he is a knave.”. Then R says: “Q is lying.”. What are Q and R?

The given information tells us $q = (p = \neg p)$ and $r = \neg q$. We begin with both axioms.

$q = (p = \neg p) \land r = \neg q$ simplify first conjunct
$\equiv \neg q \land r = \neg q$ use first conjunct to simplify second conjunct
$\equiv \neg q \land r$ so Q is a knave and R is a knight.

(e) You ask P: “How many of you are knights?” P mumbles. So Q says: “P said there is exactly one knight among us.”. R says: “Q is lying.”. What are Q and R?

We start with the given information.

$q = (p = ((p \lor q \lor r) \land \neg (p \land q) \land \neg (p \land r) \land \neg (q \land r)) \land r = \neg q$ Use $r = \neg q$ with
§ The given information is the top line.

(h) P, Q, and R each say: “The other two are knaves.”. How many knaves are there?

We are given

(g) P says that Q and R are the same (both knaves or both knights). Someone asks R whether

(f) P says: “We're all knaves.”. Q says: “No, exactly one of us is a knight.”. What are P, Q, and R?

We don't know who is a knight and who is a knave, but we know that there is one knight

and two knaves.

hence Q is a knave and R is a knight.

Hence P is a knave, Q is a knight, and R is a knave.

(g) P says that Q and R are the same (both knaves or both knights). Someone asks R whether P and Q are the same. What is R’s answer?

We are given \( p = q = r \). By symmetry and associativity, that's \( r = \neg (p = q) \). So R says that P and Q are the same.

(h) P, Q, and R each say: “The other two are knaves.”. How many knaves are there?

The given information is the top line.

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\begin{align*}
\equiv & \quad q = (p = ((p \lor q \lor \neg q) \land \neg(p \land q) \land \neg(p \land \neg q) \land \neg(q \land \neg q))) \land r = \neg q \\
\equiv & \quad q = (p = (\top \land \neg(p \land q) \land \neg(p \land \neg q) \land \top)) \land r = \neg q \\
\equiv & \quad q = (p = \neg((p \land q) \lor (p \land \neg q))) \land r = \neg q \\
\equiv & \quad q = (p = \neg p) \land r = \neg q \\
\equiv & \quad \neg q \land r = \neg q \\
\equiv & \quad r \land \neg q
\end{align*}
\]

\[
\begin{align*}
\equiv & \quad p = (\neg p \land \neg q \land \neg r) \land q = (\neg p \land \neg q) \land r = (\neg p \land \neg q) \\
\equiv & \quad (p \land \neg q \land \neg r \lor \neg p \land \neg(q \land \neg r)) \land q = (\neg p \land \neg r) \land r = (\neg p \land \neg q) \\
\equiv & \quad (p \land \neg q \land \neg r \lor \neg p \land (q \lor r)) \land q = (\neg p \land \neg r) \land r = (\neg p \land \neg q) \\
\equiv & \quad (p \land \neg q \land \neg r \lor \neg p \land q \lor \neg p \land r) \land q = (\neg p \land \neg r) \land r = (\neg p \land \neg q) \\
\equiv & \quad (p \land \neg q \land \neg r \lor \neg p \land q \land \neg p \land \neg r \lor \neg p \land \neg q \land r) \\
\equiv & \quad (p \land \neg q \land \neg r \lor \neg p \land q \land \neg p \land \neg r \lor \neg p \land \neg q \land r) \\
\equiv & \quad (p \land \neg q \land \neg r \lor \neg p \land q \land \neg r \lor \neg p \land \neg q \land r) \\
\Rightarrow & \quad p \land \neg q \land \neg r \lor \neg p \land q \land \neg r \lor \neg p \land \neg q \land r
\end{align*}
\]

We don't know who is a knight and who is a knave, but we know that there is one knight
and two knaves.