254 (alternate Ackermann) For each of the following functions f, refine $n:= f\ m\ n$, find a time bound (possibly involving f), and find a space bound.

(a)
$$f\ 0\ n\ =\ n+2$$
$$f\ 1\ 0\ =\ 0$$
$$f\ (m+2)\ 0\ =\ 1$$
$$f\ (m+1)\ (n+1)\ =\ f\ m\ (f\ (m+1)\ n)$$

(b)
$$f\ 0\ n\ =\ n\times 2$$
$$f\ (m+1)\ 0\ =\ 1$$
$$f\ (m+1)\ (n+1)\ =\ f\ m\ (f\ (m+1)\ n)$$

(c)
$$f\ 0\ n\ =\ n+1$$
$$f\ 1\ 0\ =\ 2$$
$$f\ 2\ 0\ =\ 0$$
$$f\ (m+3)\ 0\ =\ 1$$
$$f\ (m+1)\ (n+1)\ =\ f\ m\ (f\ (m+1)\ n)$$

After trying the question, scroll down to the solution.

§ Part (a) defines the same function *ack* as in Exercise 253. Parts (b) and (c) do not, but they define very similar, very closely related functions.

(a)

$$f\,0\,n \;=\; n+2$$
$$f\,1\,0 \;=\; 0$$
$$f\,(m+2)\,0 \;=\; 1$$
$$f\,(m+1)\,(n+1) \;=\; f\,m\,(f\,(m+1)\,n)$$

§ $n:= f\,m\,n \;\Leftarrow$
if $m=0$ **then** $n:= n+2$
else if $m=1 \wedge n=0$ **then** $n:= 0$
else if $n=0$ **then** $n:= 1$
else $n:= n-1.\; n:= f\,m\,n.\; m:= m-1.\; n:= f\,m\,n.\; m:= m+1$ **fi fi fi**

For a time bound, we want a function g such that

$t' \le t + g\,m\,n \;\wedge\; n' = f\,m\,n \;\wedge\; m'=m \;\Leftarrow$
if $m=0$ **then** $n:= n+2$
else if $m=1 \wedge n=0$ **then** $n:= 0$
else if $n=0$ **then** $n:= 1$
else $n:= n-1.\; t:= t+1.\; t' \le t + g\,m\,n \;\wedge\; n' = f\,m\,n \;\wedge\; m'=m.$
$m:= m-1.\; t' \le t + g\,m\,n \;\wedge\; n' = f\,m\,n \;\wedge\; m'=m.\; m:= m+1$
fi fi fi

In the last alternative, I put $t:= t+1$ before the first recursive call, but not before the second. The one occurrence ensures that every loop includes a time increment. But I could have put another one in. Using Refinement by Cases, and throwing away the unnecessary pieces, we need f to satisfy four things.

$$t' \le t + g\,m\,n \;\Leftarrow\; m=0 \wedge t'=t$$
$$t' \le t + g\,m\,n \;\Leftarrow\; m=1 \wedge n=0 \wedge t'=t$$
$$t' \le t + g\,m\,n \;\Leftarrow\; m>1 \wedge n=0 \wedge t'=t$$
$$t' \le t + g\,m\,n \;\Leftarrow\; m>0 \wedge n>0 \wedge t' \le t + 1 + g\,m\,(n-1) + g\,(m-1)\,(f\,m\,(n-1))$$

Simplifying,

$$g\,0\,n \;\ge\; 0$$
$$g\,m\,0 \;\ge\; 0$$
$$g\,(m+1)\,(n+1) \;\ge\; g\,(m+1)\,n + g\,m\,(f\,(m+1)\,n) + 1$$

These are the constraints on g. So replace $\ge$ by $=$ and we have a definition of g that gives the exact execution time (in terms of f).
SPACE BOUND NOT DONE YET

(b)

$$f\,0\,n \;=\; n\times 2$$
$$f\,(m+1)\,0 \;=\; 1$$
$$f\,(m+1)\,(n+1) \;=\; f\,m\,(f\,(m+1)\,n)$$

not done

(c)

$$f\,0\,n \;=\; n+1$$
$$f\,1\,0 \;=\; 2$$
$$f\,2\,0 \;=\; 0$$
$$f\,(m+3)\,0 \;=\; 1$$
$$f\,(m+1)\,(n+1) \;=\; f\,m\,(f\,(m+1)\,n)$$

not done