For each of the following functions $f$, refine $n := f m n$, find a time bound (possibly involving $f$), and find a space bound.

§ Part (a) defines the same function $\text{ack}$ as in Exercise 253. Parts (b) and (c) do not, but they define very similar, very closely related functions.

(a) $f 0 n = n + 2$
$f 1 0 = 0$
$f (m+2) 0 = 1$
$f (m+1) (n+1) = f m (f (m+1) n)$

§

n := f m n  \iff

if $m = 0$ then $n := n + 2$
else if $m = 1 \land n = 0$ then $n := 0$
else if $n = 0$ then $n := 1$
else $n := n-1$. $n := f m n$. $m := m-1$. $n := f m n$. $m := m+1$

For a time bound, we want a function $g$ such that

$t' \leq t + g m n \land n' = f m n \land m' = m$

$\iff$

if $m = 0$ then $n := n + 2$
else if $m = 1 \land n = 0$ then $n := 0$
else if $n = 0$ then $n := 1$
else $n := n-1$. $t := t+1$. $t' \leq t + g m n \land n' = f m n \land m' = m$. $m := m-1$. $t' \leq t + g m n \land n' = f m n \land m' = m$. $m := m+1$

In the last alternative, I put $t := t+1$ before the first recursive call, but not before the second. The one occurrence ensures that every loop includes a time increment. But I could have put another one in. Using Refinement by Cases, and throwing away the unnecessary pieces, we need $f$ to satisfy four things.

$t' \leq t + g m n \iff m = 0 \land t' = t$
$t' \leq t + g m n \iff m = 1 \land n = 0 \land t' = t$
$t' \leq t + g m n \iff m > 1 \land n = 0 \land t' = t$
$t' \leq t + g m n \iff m > 0 \land n > 0 \land t' \leq t + 1 + g m (n-1) + g (m-1) (f m (n-1))$

Simplifying,

$g 0 n \geq 0$
$g m 0 \geq 0$
$g (m+1) (n+1) \geq g (m+1) n + g m (f (m+1) n) + 1$

These are the constraints on $g$. So replace $\geq$ by $=$ and we have a definition of $g$ that gives the exact execution time (in terms of $f$).

SPACE BOUND NOT DONE YET

(b) $f 0 n = n \times 2$
$f (m+1) 0 = 1$
$f (m+1) (n+1) = f m (f (m+1) n)$

(c) $f 0 n = n + 1$
$f 1 0 = 2$
$f 2 0 = 0$
$f (m+3) 0 = 1$
$f (m+1) (n+1) = f m (f (m+1) n)$