For each of the following functions \( f \), refine \( n := f \ m \ n \), find a time bound (possibly involving \( f \)), and find a space bound.

\( n := f \ m \ n \) \hspace{1cm} \text{if} m = 0 \text{ then } n := n + 2 \hspace{1cm} \text{else if} m = 1 \land n = 0 \text{ then } n := 0 \hspace{1cm} \text{else if} n = 0 \text{ then } n := 1 \hspace{1cm} \text{else } n := n - 1. \)

\( n := f \ m \ n \) \hspace{1cm} \text{for a time bound, we want a function } g \text{ such that} \hspace{1cm} m = 0 \text{ then } n := n + 2 \hspace{1cm} \text{else if} m = 1 \land n = 0 \text{ then } n := 0 \hspace{1cm} \text{else if} n = 0 \text{ then } n := 1 \hspace{1cm} \text{else } n := n - 1. \)

\( t := t + 1 \). \hspace{1cm} \text{In the last alternative, I put } t := t + 1 \text{ before the first recursive call, but not before the second. The one occurrence ensures that every loop includes a time increment. But I could have put another one in. Using Refinement by Cases, and throwing away the unnecessary pieces, we need } f \text{ to satisfy four things.} \hspace{1cm} m = 0 \land t' = t \hspace{1cm} \text{else if} m = 1 \land n = 0 \land t' = t \hspace{1cm} \text{else if } n = 0 \text{ then } n := 1 \hspace{1cm} \text{else } n := n - 1. \)

\( m = m - 1. \hspace{1cm} t' \leq t + g \ m \ n \land n' = f \ m \ n \land m' = m. \hspace{1cm} t := t + g \ m \ n \land m' = m. \)

\( m := m + 1. \hspace{1cm} t' \leq t + g \ m \ n \land n' = f \ m \ n \land m' = m. \hspace{1cm} m := m + 1. \)

Simplifying,
\( g \ 0 \ n \geq 0 \hspace{1cm} g \ 0 \ n \geq 0 \hspace{1cm} g \ (m+1) \ (n+1) \geq g \ (m+1) \ n + g \ m \ (f \ (m+1) \ n) + 1 \)

These are the constraints on \( g \). So replace \( \geq \) by \( = \) and we have a definition of \( g \) that gives the exact execution time (in terms of \( f \)).

SPACE BOUND NOT DONE YET

\( f \ 0 \ n = n \times 2 \hspace{1cm} f \ 1 \ 0 = 2 \hspace{1cm} f \ 2 \ 0 = 0 \hspace{1cm} f \ m+3 \ 0 = 1 \hspace{1cm} f \ 0 \ n = n + 1 \hspace{1cm} f \ 1 \ 0 = 2 \hspace{1cm} f \ 2 \ 0 = 0 \hspace{1cm} f \ m+3 \ 0 = 1 \)