(transitive closure) A relation \( R: (0,..,n) \rightarrow (0,..,n) \rightarrow \text{bin} \) can be represented by a square binary array of size \( n \). Given a relation in the form of a square binary array, write a program to find

(a) its transitive closure (the strongest transitive relation that is implied by the given relation).

§ Let \( P_{ijk} \) mean “there is a path in \( R \) from \( j \) to \( k \) via zero or more intermediate nodes all of which are less than \( i \)”. Formally,

\[
P_0 = R \\
\forall i, j, k: P(i+1)_{jk} = P_{ijk} \lor P_{ij} \land P_{ik}
\]

Then we can say that \( R' \) is the transitive closure of \( R \) as follows:

\[
R' = P_n
\]

This is just right for a for-loop (Chapter 5) with invariant \( R = Pi \).

\[
R = P 0 \Rightarrow R' = P n \iff \text{for } i:= 0;..,n \text{ do } R = P i \Rightarrow R' = P(i+1) \text{ od}
\]

\[
\text{for } j:= 0;..,n \text{ do for } k:= 0;..,n \text{ do } R := (j,k) \Rightarrow R_{jk} \lor R_{ji} \land R_{ik} \mid R \text{ od od}
\]

That's the whole thing. If you want more detail, define \( A \) as follows.

\[
A_{ijk} = (\forall r: 0,..,n \forall c: 0,..,n: R_{rc} = P(i+1)r c) \\
\land (\forall c: 0,..,k: R_{jc} = P(i+1)j c) \\
\land (\forall c: k,..,n: R_{jc} = P i j c) \\
\land (\forall r: j+1,..,n \forall c: k,..,n: R_{rc} = P i r c)
\]

Now \( A_{000} = R = Pi \) and \( A_{ijn} = A_{i(j+1)0} \) and \( A_{in0} = A_{i+1}00 \).

\[
A_{000} \Rightarrow A'n 00 \iff \text{for } i:= 0;..,n \text{ do } A_{i00} \Rightarrow A'(i+1)00 \text{ od}
\]

\[
A_{ij0} \Rightarrow A'(i+1)00 \iff \text{for } j:= 0;..,n \text{ do } A_{ij0} \Rightarrow A'(i+1)00 \text{ od}
\]

\[
A_{ijk} \Rightarrow A'(i+1)00 \iff \text{for } k:= 0;..,n \text{ do } A_{ijk} \Rightarrow A'i_{j(k+1)} \text{ od}
\]

\[
A_{ijk} \Rightarrow A'i_{j(k+1)} \iff R := (j,k) \Rightarrow R_{jk} \lor R_{ji} \land R_{ik}
\]

Of course, for-loops are not necessary.

(b) its reflexive transitive closure (the strongest reflexive and transitive relation that is implied by the given relation).

§ This is similar to part (a), but this time we define \( P 00k = j=k \lor R_{jk} \). Since \( P 00 \) is not true initially, we need to start with

\[
R' = P n \iff \text{for } j:= 0;..,n \text{ do } R := (j,j) \Rightarrow T \mid R \text{ od.}
\]

\[
R = P 00 \Rightarrow R' = P n
\]

and then continue as before.