248 (transitive closure) A relation R: $(0,..n)\to(0,..n)\to bin$ can be represented by a square binary array of size n . Given a relation in the form of a square binary array, write a program to find

(a) its transitive closure (the strongest transitive relation implied by the given relation).

(b) its reflexive transitive closure (the strongest reflexive and transitive relation that is implied by the given relation).

After trying the question, scroll down to the solution.

(a) its transitive closure (the strongest transitive relation that is implied by the given relation).

§ Let $P\ i\ j\ k$ mean "there is a path in R from j to k via zero or more intermediate nodes all of which are less than i". Formally,

$$P\,0 = R$$
$$\forall i, j, k\cdot\ P(i{+}1)j\ k \ \ = \ \ P\ i\ j\ k\ \vee\ P\ i\ j\ i \wedge P\ i\ i\ k$$

Then we can say that R' is the transitive closure of R as follows:

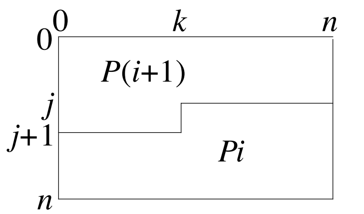
$$R' = P\ n$$

This is just right for a **for**-loop (Chapter 5) with invariant $R = P\ i$.

$$R = P\,0\ \Rightarrow\ R' = P\ n\ \ \Leftarrow\ \ \textbf{for}\ i{:=}\ 0;..n\ \textbf{do}\ R = P\ i\ \Rightarrow\ R'{=}P(i{+}1)\ \textbf{od}$$
$$R = P\ i\ \Rightarrow\ R'{=}P(i{+}1)\ \ \Leftarrow$$
$$\textbf{for}\ j{:=}\ 0;..n\ \textbf{do for}\ k{:=}\ 0;..n\ \textbf{do}\ R{:=}\ (j;k) \to (R\ j\ k\ \vee\ R\ j\ i \wedge R\ i\ k)\ |\ R\ \textbf{od od}$$

That's the whole thing. If you want more detail, define A as follows.

$$\begin{aligned} A\ i\ j\ k\ \ =\ \ & (\forall r{:}\ 0,..j\cdot\ \forall c{:}\ 0,..n\cdot\ R\ r\ c = P(i{+}1)r\ c) \\ \wedge\ & (\forall c{:}\ 0,..k\cdot\ R\ j\ c = P(i{+}1)j\ c) \\ \wedge\ & (\forall c{:}\ k,..n\cdot\ R\ j\ c = P\ i\ j\ c) \\ \wedge\ & (\forall r{:}\ j{+}1,..n\cdot\ \forall c{:}\ k,..n\cdot\ R\ r\ c = P\ i\ r\ c) \end{aligned}$$



Now $A\ i\,0\,0\ =\ R = P\ i$ and $A\ i\ j\ n\ =\ A\ i(j{+}1)0$ and $A\ i\ n\,0\ =\ A(i{+}1)0\,0$.

$$A\,0\,0\,0 \Rightarrow A'n\,0\,0\ \ \Leftarrow\ \ \textbf{for}\ i{:=}\ 0;..n\ \textbf{do}\ A\ i\,0\,0 \Rightarrow A'(i{+}1)0\,0\ \textbf{od}$$
$$A\ i\,0\,0 \Rightarrow A'(i{+}1)0\,0\ \ \Leftarrow\ \ \textbf{for}\ j{:=}\ 0;..n\ \textbf{do}\ A\ i\ j\,0 \Rightarrow A'i(j{+}1)0\ \textbf{od}$$
$$A\ i\ j\,0 \Rightarrow A'i(j{+}1)0\ \ \Leftarrow\ \ \textbf{for}\ k{:=}\ 0;..n\ \textbf{do}\ A\ i\ j\ k \Rightarrow A'i\ j(k{+}1)\ \textbf{od}$$
$$A\ i\ j\ k \Rightarrow A'i\ j(k{+}1)\ \ \Leftarrow\ \ R{:=}\ (j;k) \to (R\ j\ k\ \vee\ R\ j\ i \wedge R\ i\ k)$$

Of course, **for**-loops are not necessary.

(b) its reflexive transitive closure (the strongest reflexive and transitive relation that is implied by the given relation).

§ This is similar to part (a), but this time we define $P\,0\ j\ k\ =\ j{=}k \vee R\ j\ k$. Since $P\,0$ is not true initially, we need to start with

$$R' = P\ n\ \ \Leftarrow\ \ \textbf{for}\ j{:=}\ 0;..n\ \textbf{do}\ R{:=}\ (j;j) \to \top\ |\ R\ \textbf{od}.$$
$$R = P\,0 \Rightarrow R' = P\ n$$

and then continue as before.