240 (minimum difference) Given two nonempty sorted lists of numbers, write a program to find a pair of items, one from each list, whose absolute difference is smallest.

After trying the question, scroll down to the solution.

§ Let the lists be L and M, and indicate the two items by the final values of index variables l: 0,..#L and m: 0,..#M. The specification is R, defined by

R = find the minimum difference items in nonempty lists L and M
= $abs\ (L\ l' - M\ m') = \Downarrow i: 0,..\#L\cdot \Downarrow j: 0,..\#M\cdot abs\ (L\ i - M\ j)$
= $\forall i: 0,..\#L\cdot \forall j: 0,..\#M\cdot abs\ (L\ l' - M\ m') \le abs\ (L\ i - M\ j)$

We also need natural variables p and q as indexes. Here are the other specifications.

Q = find the minimum difference items in nonempty lists $L[p;..\#L]$ and $M[q;..\#M]$
= $p<\#L \land q<\#M$
$\Rightarrow abs\ (L\ l' - M\ m') = \Downarrow i: p,..\#L\cdot \Downarrow j: q,..\#M\cdot abs\ (L\ i - M\ j)$
= $p<\#L \land q<\#M$
$\Rightarrow \forall i: p,..\#L\cdot \forall j: q,..\#M\cdot abs\ (L\ l' - M\ m') \le abs\ (L\ i - M\ j)$

P = $p<\#L \land q<\#M \land abs\ (L\ l - M\ m) = d \le abs\ (L\ p - M\ q)$
$\Rightarrow abs(L\ l' - M\ m')\le d \land \forall i: p,..\#L\cdot \forall j: q,..\#M\cdot abs(L\ l'-M\ m')\le abs(L\ i-M\ j)$

N = $p<\#L \land q<\#M \land abs\ (L\ l - M\ m) = d$
$\Rightarrow abs(L\ l'-M\ m')\le d \land \forall i: p,..\#L\cdot \forall j: q,..\#M\cdot abs(L\ l'-M\ m')\le abs(L\ i-M\ j)$

Now we refine as follows.

R ⇐ p:= 0. q:= 0. Q
Q ⇐ l:= p. m:= q. d:= $abs\ (L\ l - M\ m)$. P
P ⇐ **if** $L\ p < M\ q$ **then** p:= p+1. **if** p=#L **then** ok **else** N **fi**
else if $L\ p > M\ q$ **then** q:= q+1. **if** q=#M **then** ok **else** N **fi**
else l:= p. m:= q **fi fi**
N ⇐ **if** $abs\ (L\ p - M\ q) < d$ **then** Q **else** P **fi**

Proof of R refinement, starting with the right side:

p:= 0. q:= 0. Q — expand Q and use substitution law twice
= R

Proof of Q refinement, starting with the right side:

l:= p. m:= q. d:= $abs\ (L\ l - M\ m)$. P — expand P and use substitution law 3 times
= $p<\#L \land q<\#M \land abs\ (L\ p - M\ q) = abs\ (L\ p - M\ q) \le abs\ (L\ p - M\ q)$
$\Rightarrow abs\ (L\ l' - M\ m') \le abs\ (L\ p - M\ q)$
$\land \forall i: p,..\#L\cdot \forall j: q,..\#M\cdot abs\ (L\ l' - M\ m') \le abs\ (L\ i - M\ j)$
— simplify antecedent; one conjunct of the consequent is a special case of the universal quantification
= $p<\#L \land q<\#M \Rightarrow \forall i: p,..\#L\cdot \forall j: q,..\#M\cdot abs\ (L\ l' - M\ m') \le abs\ (L\ i - M\ j)$
= Q

The proof of the P refinement can be broken into five cases.

(a) $P \Leftarrow L\ p < M\ q \land (p:= p+1.\ p=\#L \land ok)$
(b) $P \Leftarrow L\ p < M\ q \land (p:= p+1.\ p\ne\#L \land N)$
(c) $P \Leftarrow L\ p > M\ q \land (q:= q+1.\ q=\#M \land ok)$
(d) $P \Leftarrow L\ p > M\ q \land (q:= q+1.\ q\ne\#M \land N)$
(e) $P \Leftarrow L\ p = M\ q \land (l:= p.\ m:= q)$

Here is the proof of case (a):

$(P \Leftarrow L\ p < M\ q \land (p:= p+1.\ p=\#L \land ok))$ — expand ok and substitution law; turn the main implication around; expand P; portation
= $L\ p < M\ q \land p+1=\#L \land l'=l \land m'=m \land p'=p+1 \land q'=q$
$\land p<\#L \land q<\#M \land abs\ (L\ l - M\ m) = d \le abs\ (L\ p - M\ q)$
$\Rightarrow abs\ (L\ l' - M\ m') \le d \land \forall i: p,..\#L\cdot \forall j: q,..\#M\cdot abs(L\ l'-M\ m') \le abs(L\ i - M\ j)$
— pick off the easy parts, and note that $\forall i: p,..\#L$ has only one domain value
⇐ $L(\#L-1) < M\ q \land q<\#M \land d \le abs\ (L(\#L-1)-M\ q)$
$\Rightarrow \forall j: q,..\#M\cdot d \le abs\ (L(\#L-1)-M\ j)$

Since $L(\#L-1) < M\ q$, therefore $L(\#L-1) - M\ q$ is negative, so for all $j\ge q$,

$L(\#L-1) - M\,j$ is at least as negative (remember that M is sorted), so $abs\ (L(\#L-1) - M\,q) \le abs\ (L(\#L-1) - M\,j)$, so by a connection law

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Here is the proof of case (b):

$(P \Leftarrow L\,p < M\,q \wedge (p{:=}\ p+1.\ \ p\ne\#L \wedge N))$ expand N and substitution law; turn the main implication around; expand P ; portation

$=\quad L\,p < M\,q \wedge p+1\ne\#L$
$\quad \wedge\ (\quad p+1<\#L\ \wedge\ q<\#M\ \wedge\ abs(L\,l - M\,m) = d$
$\qquad \Rightarrow\quad abs(L\,l' - M\,m') \le d$
$\qquad\quad \wedge\ \ \forall i{:}\ p+1,..\#L\cdot\ \forall j{:}\ q,..\#M\cdot\ abs(L\,l' - M\,m') \le abs(L\,i - M\,j))$
$\quad \wedge\ \ p<\#L\ \wedge\ q<\#M\ \wedge\ abs(L\,l - M\,m) = d \le abs(L\,p - M\,q)$
$\Rightarrow\ abs(L\,l' - M\,m') \le d\ \wedge \forall i{:}\ p,..\#L\cdot\ \forall j{:}\ q,..\#M\cdot\ abs(L\,l' - M\,m') \le abs(L\,i - M\,j)$

discharge

$=\quad L\,p < M\,q \wedge p+1<\#L\ \wedge\ q<\#M$
$\quad \wedge\ \ abs(L\,l' - M\,m') \le d$
$\quad \wedge\ \ (\forall i{:}\ p+1,..\#L\cdot\ \forall j{:}\ q,..\#M\cdot\ abs(L\,l' - M\,m') \le abs(L\,i - M\,j))$
$\quad \wedge\ \ abs(L\,l - M\,m) = d \le abs(L\,p - M\,q)$
$\Rightarrow\ abs(L\,l' - M\,m') \le d\ \wedge \forall i{:}\ p,..\#L\cdot\ \forall j{:}\ q,..\#M\cdot\ abs(L\,l' - M\,m') \le abs(L\,i - M\,j)$

In the antecedent, we have the first conjunct of the consequent. We also have the second conjunct except only for $i{=}p$.

$\Leftarrow\quad L\,p < M\,q \wedge p+1<\#L\ \wedge\ q<\#M$
$\quad \wedge\ \ abs(L\,l' - M\,m') \le d$
$\quad \wedge\ \ (\forall i{:}\ p+1,..\#L\cdot\ \forall j{:}\ q,..\#M\cdot\ abs(L\,l' - M\,m') \le abs(L\,i - M\,j))$
$\quad \wedge\ \ abs(L\,l - M\,m) = d \le abs(L\,p - M\,q)$
$\Rightarrow\ \forall j{:}\ q,..\#M\cdot\ abs(L\,l' - M\,m') \le abs(L\,p - M\,j)$

From the antecedent, $abs(L\,l' - M\,m') \le d$, and $d \le abs(L\,p - Mq)$, so $abs(L\,l' - M\,m') \le abs(L\,p - M\,q)$. Also, $L\,p < M\,q$, and M is sorted, so if $q{\le}j$, then $abs(L\,p - M\,q) \le abs(L\,p - M\,j)$. And so $abs(L\,l' - M\,m') \le abs(L\,p - M\,j)$.

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The proof of case (c) is just like the proof of case (a).
The proof of case (d) is just like the proof of case (b).
Here is the proof of case (e):

$(P \Leftarrow L\,p = M\,q \wedge (l{:=}\ p.\ \ m{:=}\ q))$ expand $m{:=}\ q$ and substitution law; turn the main implication around; expand P ; portation

$=\quad L\,p = M\,q \wedge l'{=}p \wedge m'{=}q \wedge p'{=}p \wedge q'{=}q$
$\quad \wedge\ p<\#L\ \wedge\ q<\#M\ \wedge\ abs(L\,l - M\,m) = d \le abs(L\,p - M\,q)$
$\Rightarrow\ abs(L\,l' - M\,m') \le d\ \wedge \forall i{:}\ p,..\#L\cdot\ \forall j{:}\ q,..\#M\cdot\ abs(L\,l' - M\,m') \le abs(L\,i - M\,j)$

from the antecedent, $abs(L\,l' - M\,m') = 0$

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Proof of N refinement, by cases, starting with the first case:

$(N \Leftarrow abs(L\,p - M\,q) < d \wedge Q)$

turn the main implication around; expand N and Q ; portation

$=\quad abs(L\,p - M\,q) < d$
$\quad \wedge\ (\quad p<\#L\ \wedge\ q<\#M$
$\qquad \Rightarrow\ \forall i{:}\ p,..\#L\cdot\ \forall j{:}\ q,..\#M\cdot\ abs(L\,l' - M\,m') \le abs(L\,i - M\,j))$
$\quad \wedge\ p<\#L\ \wedge\ q<\#M\ \wedge\ abs(L\,l - M\,m) = d$
$\Rightarrow\ abs(L\,l' - M\,m') \le d\ \wedge \forall i{:}\ p,..\#L\cdot\ \forall j{:}\ q,..\#M\cdot\ abs(L\,l' - M\,m') \le abs(L\,i - M\,j)$

discharge

$=\quad abs(L\,p - M\,q) < d$
$\quad \wedge\ (\forall i{:}\ p,..\#L\cdot\ \forall j{:}\ q,..\#M\cdot\ abs(L\,l' - M\,m') \le abs(L\,i - M\,j))$
$\quad \wedge\ p<\#L\ \wedge\ q<\#M\ \wedge\ abs(L\,l - M\,m) = d$

$\Rightarrow\ abs(L\ l' - M\ m') \le d\ \wedge \forall i{:}\ p,..\#L\cdot \forall j{:}\ q,..\#M\cdot abs(L\ l' - M\ m') \le abs(L\ i - M\ j)$

The second conjunct of the antecedent is also the second conjunct of the consequent. So that leaves

$\Leftarrow\quad abs(L\ p - M\ q) < d$
$\wedge\ (\forall i{:}\ p,..\#L\cdot \forall j{:}\ q,..\#M\cdot abs(L\ l' - M\ m') \le abs(L\ i - M\ j))$
$\wedge\ p<\#L\ \wedge\ q<\#M\ \wedge\ abs(L\ l - M\ m) = d$
$\Rightarrow\ abs(L\ l' - M\ m') \le d$

Specialize the quantification, and drop some other parts of the antecedent

$\Leftarrow\quad abs(L\ p - M\ q) < d$
$\wedge\ abs(L\ l' - M\ m') \le abs(L\ p - M\ q)$
$\Rightarrow\ abs(L\ l' - M\ m') \le d$

transitivity

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Proof of N refinement, last case:

$(N \Leftarrow abs(L\ p - M\ q) \ge d \wedge P)$

turn the main implication around; expand N and P ; portation

$=\quad abs(L\ p - M\ q) \ge d$
$\wedge\ (\quad p<\#L\ \wedge\ q<\#M\ \wedge\ abs(L\ l - M\ m) = d \le abs(L\ p - M\ q)$
$\quad \Rightarrow\ abs(L\ l' - M\ m') \le d$
$\quad\quad \wedge\ \forall i{:}\ p,..\#L\cdot \forall j{:}\ q,..\#M\cdot abs(L\ l' - M\ m') \le abs(L\ i - M\ j))$
$\wedge\ p<\#L\ \wedge\ q<\#M\ \wedge\ abs(L\ l - M\ m) = d$
$\Rightarrow\ abs(L\ l' - M\ m') \le d\ \wedge \forall i{:}\ p,..\#L\cdot \forall j{:}\ q,..\#M\cdot abs(L\ l' - M\ m') \le abs(L\ i - M\ j)$

discharge

$=\quad abs(L\ p - M\ q) \ge d$
$\wedge\ abs(L\ l' - M\ m') \le d$
$\wedge\ (\forall i{:}\ p,..\#L\cdot \forall j{:}\ q,..\#M\cdot abs(L\ l' - M\ m') \le abs(L\ i - M\ j))$
$\wedge\ p<\#L\ \wedge\ q<\#M \wedge abs(L\ l - M\ m) = d$
$\Rightarrow\ abs(L\ l' - M\ m') \le d\ \wedge \forall i{:}\ p,..\#L\cdot \forall j{:}\ q,..\#M\cdot abs(L\ l' - M\ m') \le abs(L\ i - M\ j)$

specialization

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For timing, we replace R by $t' \le t+\#L+\#M$ and the other three by $t' \le t+\#L-p+\#M-q$.