239 Let L and M be sorted lists of numbers. Write a program to find the number of pairs of indexes i: $\square L$ and j: $\square M$ such that $L\, i \le M\, j$.

After trying the question, scroll down to the solution.

§ The answer will be reported as the final value of natural variable n . The specification is S , defined as

$$S \;=\; n' \;=\; \Sigma i\colon \Box L\cdot\ \text{¢}\S j\colon \Box M\cdot\ L\,i \le M\,j$$

I need index variables, one for each list, so let them be l and m . Define specification R as

$$R \;=\; 0{\le}l{\le}\#L \wedge 0{\le}m{\le}\#M \;\;\Rightarrow\;\; n' \;=\; n + \Sigma i\colon l,..\#L\cdot\ \text{¢}\S j\colon m,..\#M\cdot\ L\,i \le M\,j$$

Perhaps it is unnecessary to write the antecedent in R explicitly because it is implicit in the use of $l,..\#L$ and $m,..\#M$. The refinements are

$$S \;\Leftarrow\; n{:=}\,0.\;\; l{:=}\,0.\;\; m{:=}\,0.\;\; R$$

$R \;\Leftarrow\;$ **if** $l{=}\#L \vee m{=}\#M$ **then** ok
else if $L\,l \le M\,m$ **then** $n{:=}\,n{+}\#M{-}m.\;\; l{:=}\,l{+}1.\;\; R$
else $m{:=}\,m{+}1.\;\; R$ **fi fi**

Proof of the S refinement:

$n{:=}\,0.\;\; l{:=}\,0.\;\; m{:=}\,0.\;\; R$ — replace R

$=\;\; n{:=}\,0.\; l{:=}\,0.\; m{:=}\,0.\; 0{\le}l{\le}\#L \wedge 0{\le}m{\le}\#M \;\Rightarrow\; n' = n + \Sigma i\colon l,..\#L\cdot\ \text{¢}\S j\colon m,..\#M\cdot\ L\,i \le M\,j$ — Substitution Law three times

$=\;\; 0{\le}0{\le}\#L \wedge 0{\le}0{\le}\#M \;\Rightarrow\; n' = 0 + \Sigma i\colon \Box L\cdot\ \text{¢}\S j\colon \Box M\cdot\ L\,i \le M\,j$

$=\;\; S$

The R refinement proof is in three cases. First case:

$(l{=}\#L \vee m{=}\#M) \wedge ok$ — replace ok

$=\;\; (l{=}\#L \vee m{=}\#M) \wedge n'{=}n \wedge l'{=}l \wedge m'{=}m$ — If $l{=}\#L$ then $\Sigma i\colon l,..\#L\cdot$ ... is a sum over an empty domain, which is 0 . If $m{=}\#M$ then $\S j\colon m,..\#M\cdot$ ... is an empty bunch, whose size is 0 . Either way, $R = (0{\le}l{\le}\#L \wedge 0{\le}m{\le}\#M \;\;\Rightarrow\;\; n'{=}n)$.

$\Rightarrow\;\; R$

The middle case:

$R \;\Leftarrow\; (l{\ne}\#L \wedge m{\ne}\#M \wedge L\,l \le M\,m \wedge (n{:=}\,n{+}\#M{-}m.\;\; l{:=}\,l{+}1.\;\; R))$ — expand R

$=\;\; (0{\le}l{\le}\#L \wedge 0{\le}m{\le}\#M \;\;\Rightarrow\;\; n' \;=\; n + \Sigma i\colon l,..\#L\cdot\ \text{¢}\S j\colon m,..\#M\cdot\ L\,i \le M\,j)$
$\Leftarrow\;\; l{\ne}\#L \wedge m{\ne}\#M \wedge L\,l \le M\,m$
$\wedge\; (\; n{:=}\,n{+}\#M{-}m.\; l{:=}\,l{+}1.$
$0{\le}l{\le}\#L \wedge 0{\le}m{\le}\#M \Rightarrow n' = n + \Sigma i\colon l,..\#L\cdot\ \text{¢}\S j\colon m,..\#M\cdot\ L\,i \le M\,j)$ — portation

$=\;\; n' \;=\; n + (\Sigma i\colon l,..\#L\cdot\ \text{¢}\S j\colon m,..\#M\cdot\ L\,i \le M\,j)$
$\Leftarrow\;\; 0{\le}l{\le}\#L \wedge 0{\le}m{\le}\#M \wedge l{\ne}\#L \wedge m{\ne}\#M \wedge L\,l \le M\,m$
$\wedge\; (\; n{:=}\,n{+}\#M{-}m.\; l{:=}\,l{+}1.$
$0{\le}l{\le}\#L \wedge 0{\le}m{\le}\#M \Rightarrow n' = n + \Sigma i\colon l,..\#L\cdot\ \text{¢}\S j\colon m,..\#M\cdot\ L\,i \le M\,j)$ — simplify and Substitution Law

$=\;\; 0{\le}l{<}\#L \wedge 0{\le}m{<}\#M \wedge L\,l \le M\,m$
$\wedge\; (\;\; 0{\le}l{+}1{\le}\#L \wedge 0{\le}m{\le}\#M$
$\Rightarrow n' = n + \#M - m + \Sigma i\colon l{+}1,..\#L\cdot\ \text{¢}\S j\colon m,..\#M\cdot\ L\,i \le M\,j)$

$\Rightarrow\;\; n' \;=\; n + \Sigma i\colon l,..\#L\cdot\ \text{¢}\S j\colon m,..\#M\cdot\ L\,i \le M\,j$

Since $0 \le l < \#L \Rightarrow 0 \le l+1 \le \#L$ and $0 \le m < \#M \Rightarrow 0 \le m \le \#M$
we can use discharge

$=\quad 0 \le l < \#L \land 0 \le m < \#M \land L\,l \le M\,m$
$\land\ n' = n + \#M - m + \Sigma i{:}\ l+1,..\#L\cdot ¢(\S j{:}\ m,..\#M\cdot L\,i \le M\,j)$
$\Rightarrow\ n' = n + \Sigma i{:}\ l,..\#L\cdot ¢\S j{:}\ m,..\#M\cdot L\,i \le M\,j$ arithmetic

$\Leftarrow\quad 0 \le l < \#L \land 0 \le m < \#M \land L\,l \le M\,m$
$\land\ \#M - m + \Sigma i{:}\ l+1,..\#L\cdot ¢(\S j{:}\ m,..\#M\cdot L\,i \le M\,j)$
$= \Sigma i{:}\ l,..\#L\cdot ¢\S j{:}\ m,..\#M\cdot L\,i \le M\,j$

$=\quad 0 \le l < \#L \land 0 \le m < \#M \land L\,l \le M\,m$
$\land\ \#M - m = ¢\S j{:}\ m,..\#M\cdot L\,l \le M\,j$
If $L\,l \le M\,m$, and M is sorted, then $\forall j{:}\ m,..\#M\cdot L\,l \le M\,j$

$=\quad \top$

Last case:

$R \Leftarrow (l \ne \#L \land m \ne \#M \land L\,l > M\,m \land (m{:}= m+1.\ R))$ expand R

$=\quad (0 \le l \le \#L \land 0 \le m \le \#M \Rightarrow n' = n + \Sigma i{:}\ l,..\#L\cdot ¢\S j{:}\ m,..\#M\cdot L\,i \le M\,j)$
$\Leftarrow\ l \ne \#L \land m \ne \#M \land L\,l > M\,m$
$\land\ (\ m{:}= m+1.$
$0 \le l \le \#L \land 0 \le m \le \#M \Rightarrow n' = n + \Sigma i{:}\ l,..\#L\cdot ¢\S j{:}\ m,..\#M\cdot L\,i \le M\,j)$
portation

$=\quad n' = n + (\Sigma i{:}\ l,..\#L\cdot ¢\S j{:}\ m,..\#M\cdot L\,i \le M\,j)$
$\Leftarrow\ 0 \le l \le \#L \land 0 \le m \le \#M \land l \ne \#L \land m \ne \#M \land L\,l > M\,m$
$\land\ (\ m{:}= m+1.$
$0 \le l \le \#L \land 0 \le m \le \#M \Rightarrow n' = n + \Sigma i{:}\ l,..\#L\cdot ¢\S j{:}\ m,..\#M\cdot L\,i \le M\,j)$
simplify and Substitution Law

$=\quad 0 \le l < \#L \land 0 \le m < \#M \land L\,l > M\,m$
$\land\ (\ 0 \le l \le \#L \land 0 \le m+1 \le \#M$
$\Rightarrow n' = n + \Sigma i{:}\ l,..\#L\cdot ¢\S j{:}\ m+1,..\#M\cdot L\,i \le M\,j)$
$\Rightarrow\ n' = n + \Sigma i{:}\ l,..\#L\cdot ¢\S j{:}\ m,..\#M\cdot L\,i \le M\,j$
Since $0 \le l < \#L \Rightarrow 0 \le l \le \#L$ and $0 \le m < \#M \Rightarrow 0 \le m+1 \le \#M$
we can use discharge

$=\quad 0 \le l < \#L \land 0 \le m < \#M \land L\,l > M\,m$
$\land\ n' = n + \Sigma i{:}\ l,..\#L\cdot ¢(\S j{:}\ m+1,..\#M\cdot L\,i \le M\,j)$
$\Rightarrow\ n' = n + \Sigma i{:}\ l,..\#L\cdot ¢\S j{:}\ m,..\#M\cdot L\,i \le M\,j$
If $L\,l > M\,m$, and L is sorted, then $\forall i{:}\ l,..\#L\cdot L\,i > M\,m$

$=\quad 0 \le l < \#L \land 0 \le m < \#M \land L\,l > M\,m$
$\land\ n' = n + \Sigma i{:}\ l,..\#L\cdot ¢(\S j{:}\ m,..\#M\cdot L\,i \le M\,j)$
$\Rightarrow\ n' = n + \Sigma i{:}\ l,..\#L\cdot ¢\S j{:}\ m,..\#M\cdot L\,i \le M\,j$ specialize

$=\quad \top$

The execution time refinements are

$t' \le t + \#L + \#M \Leftarrow$
$n{:}= 0.\ l{:}= 0.\ m{:}= 0.\ 0 \le l \le \#L \land 0 \le m \le \#M \Rightarrow t' \le t + \#L - l + \#M - m$

$0 \le l \le \#L \land 0 \le m \le \#M \Rightarrow t' \le t + \#L - l + \#M - m \Leftarrow$
if $l{=}\#L \lor m{=}\#M$ **then** ok
else if $L\,l \le M\,m$ **then** $n{:}= n+1.\ l{:}= l+1.\ t{:}= t+1.$
$0 \le l \le \#L \land 0 \le m \le \#M \Rightarrow t' \le t + \#L - l + \#M - m$
else $m{:}= m+1.\ t{:}= t+1.$
$0 \le l \le \#L \land 0 \le m \le \#M \Rightarrow t' \le t + \#L - l + \#M - m$ **fi fi**

Proof of first refinement:

$n:= 0.\ l:= 0.\ m:= 0.\ 0 \le l \le \#L \land 0 \le m \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m$

Substitution Law three times

$= \quad t' \le t + \#L + \#M$

Proof of last refinement, first case:

$(0 \le l \le \#L \land 0 \le m \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m) \ \Leftarrow\ (l = \#L \lor m = \#M) \land ok$

portation

$= \quad (l = \#L \lor m = \#M) \land ok \land (0 \le l \le \#L \land 0 \le m \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m)$

expand ok

$= \quad (l = \#L \lor m = \#M) \land n' = n \land l' = l \land m' = m \land t' = t$

$\land\ (0 \le l \le \#L \land 0 \le m \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m)$ use context $t' = t$

$= \quad (l = \#L \lor m = \#M) \land n' = n \land l' = l \land m' = m \land t' = t$

$\land\ (0 \le l \le \#L \land 0 \le m \le \#M \ \Rightarrow\ t \le t + \#L - l + \#M - m)$

$= \quad \top$

Proof of last refinement, middle case:

$(0 \le l \le \#L \land 0 \le m \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m)$

$\Leftarrow \quad l \ne \#L \land m \ne \#M \land L\,l \le M\,m$

$\land\ (n:= n+1.\ l:= l+1.\ t:= t+1.\ 0 \le l \le \#L \land 0 \le m \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m)$

Substitution Law three times, and simplify

$= \quad (0 \le l \le \#L \land 0 \le m \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m)$

$\Leftarrow \quad l \ne \#L \land m \ne \#M \land L\,l \le M\,m$

$\land\ (0 \le l+1 \le \#L \land 0 \le m \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m)$ portation

$= \quad 0 \le l \le \#L \land 0 \le m \le \#M \land l \ne \#L \land m \ne \#M \land L\,l \le M\,m$

$\land\ (0 \le l+1 \le \#L \land 0 \le m \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m)$

$\Rightarrow\ t' \le t + \#L - l + \#M - m$ simplify

$= \quad 0 \le l < \#L \land 0 \le m < \#M \land L\,l \le M\,m$

$\land\ (0 \le l+1 \le \#L \land 0 \le m \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m)$

$\Rightarrow\ t' \le t + \#L - l + \#M - m$ discharge

$= \quad 0 \le l < \#L \land 0 \le m < \#M \land Ll \le Mm \land t' \le t + \#L - l + \#M - m$

$\Rightarrow\ t' \le t + \#L - l + \#M - m$ specialize

$= \quad \top$

Proof of last refinement, last case:

$(0 \le l \le \#L \land 0 \le m \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m)$

$\Leftarrow \quad l \ne \#L \land m \ne \#M \land L\,l > M\,m$

$\land\ (m:= m+1.\ t:= t+1.\ 0 \le l \le \#L \land 0 \le m \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m)$

Substitution Law twice, and simplify

$= \quad (0 \le l \le \#L \land 0 \le m \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m)$

$\Leftarrow \quad l \ne \#L \land m \ne \#M \land L\,l \le M\,m$

$\land\ (0 \le l \le \#L \land 0 \le m+1 \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m)$ portation

$= \quad 0 \le l \le \#L \land 0 \le m \le \#M \land l \ne \#L \land m \ne \#M \land L\,l > M\,m$

$\land\ (0 \le l \le \#L \land 0 \le m+1 \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m)$

$\Rightarrow\ t' \le t + \#L - l + \#M - m$ simplify

$= \quad 0 \le l < \#L \land 0 \le m < \#M \land L\,l > M\,m$

$\land\ (0 \le l \le \#L \land 0 \le m+1 \le \#M \ \Rightarrow\ t' \le t + \#L - l + \#M - m)$

$\Rightarrow\ t' \le t + \#L - l + \#M - m$ discharge

$= \quad 0 \le l < \#L \land 0 \le m < \#M \land L\,l > M\,m \land t' \le t + \#L - l + \#M - m$

$\Rightarrow\ t' \le t + \#L - l + \#M - m$ specialize

$= \quad \top$