238 (pivot) You are given a nonempty list of positive numbers. Write a program to find the balance point, or pivot. Each item contributes its value (weight) times its distance from the pivot to its side of the balance. Item i is considered to be located at point $i + 1/2$, and the pivot point may likewise be noninteger.

After trying the question, scroll down to the solution.

§ The result we want is R where

$$R = (\Sigma i: 0,..\#L\cdot L\,i \times (p'-(i+1/2))) = 0 \;\wedge\; t' = t+\#L$$
$$= p' = (\Sigma i: 0,..\#L\cdot L\,i \times i) / (\Sigma L) + 1/2 \;\wedge\; t' = t+\#L$$

Let $Q = n' = n+(\Sigma i: j,..\#L\cdot L\,i \times i) \;\wedge\; d' = d+(\Sigma i: j,..\#L\cdot L\,i) \;\wedge\; t' = t+\#L-j$

Then the problem is solved by the following refinements.

$$R \Leftarrow j:= 0.\; n:= 0.\; d:= 0.\; Q.\; p:= n/d + 1/2$$
$$Q \Leftarrow \textbf{if } j=\#L \textbf{ then } ok$$
$$\textbf{else } n:= n + L\,j \times j.\; d:= d + L\,j.\; j:= j+1.\; t:= t+1.\; Q \textbf{ fi}$$

Here's the proof of the first refinement.

$j:= 0.\; n:= 0.\; d:= 0.\; Q.\; p:= n/d + 1/2$ — replace Q

$=$ $j:= 0.\; n:= 0.\; d:= 0.$
$n' = n+(\Sigma i: j,..\#L\cdot L\,i \times i) \;\wedge\; d' = d+(\Sigma i: j,..\#L\cdot L\,i) \;\wedge\; t' = t+\#L-j.$
$p:= n/d + 1/2$ — substitution law 3 times

$=$ $n' = (\Sigma i: 0,..\#L\cdot L\,i \times i) \;\wedge\; d' = (\Sigma i: 0,..\#L\cdot L\,i) \;\wedge\; t' = t+\#L.$
$p:= n/d + 1/2$ — sequential composition

$=$ $n' = (\Sigma i: 0,..\#L\cdot L\,i \times i) \;\wedge\; d' = (\Sigma i: 0,..\#L\cdot L\,i) \;\wedge\; t' = t+\#L.$
$p' = (\Sigma i: 0,..\#L\cdot L\,i \times i)/(\Sigma i: 0,..\#L\cdot L\,i) + 1/2$

$\Rightarrow$ R

Here's the first case of the last refinement.

$Q \Leftarrow j=\#L \wedge ok$ — expand Q and ok

$=$ $Q \Leftarrow j=\#L \wedge n'=n \wedge d'=d \wedge t'=t$

$=$ $n' = n+(\Sigma i: j,..\#L\cdot L\,i \times i) \;\wedge\; d' = d+(\Sigma i: j,..\#L\cdot L\,i) \;\wedge\; t' = t+\#L-j$
$\Leftarrow j=\#L \wedge n'=n \wedge d'=d \wedge t'=t$ — context

$=$ $n' = n+0 \;\wedge\; d' = d+0 \;\wedge\; t' = t+0 \;\Leftarrow\; j=\#L \wedge n'=n \wedge d'=d \wedge t'=t$ — specialization

$\Rightarrow$ $\top$

Last case of the last refinement:

$j\ne\#L \wedge (n:= n + L\,j \times j.\; d:= d + L\,j.\; j:= j+1.\; t:= t+1.\; Q)$
— expand Q and substitution

$=$ $j\ne\#L \;\wedge\; n' = n + L\,j \times j + (\Sigma i: j+1,..\#L\cdot L\,i \times i) \;\wedge\; d' = d + L\,j + (\Sigma i: j+1,..\#L\cdot L\,i)$
$\wedge\; t' = t+1+\#L-(j+1)$ — simplify

$=$ $j\ne\#L \;\wedge\; n' = n+(\Sigma i: j,..\#L\cdot L\,i \times i) \;\wedge\; d' = d+(\Sigma i: j,..\#L\cdot L\,i) \;\wedge\; t' = t+\#L-j$
— specialize

$\Rightarrow$ Q