221 (natural binary logarithm) The natural binary logarithm of a positive integer p is the natural number b satisfying

$$2^b \le p < 2^{b+1}$$

Write a program to find the natural binary logarithm of a given positive integer p in $log\ p$ time.

After trying the question, scroll down to the solution.

§ Let b and d be nat variables, and let p be a positive integer constant.

Define $D = d=2^b \le p \Rightarrow 2^{b'} \le p < 2^{b'+1}$

Then $2^{b'} \le p < 2^{b'+1} \Leftarrow b{:=}0.\ d{:=}1.\ D$

Proof, starting with the right side:

$b{:=}0.\ d{:=}1.\ D$ — definition of D

$= b{:=}0.\ d{:=}1.\ d=2^b \le p \Rightarrow 2^{b'} \le p < 2^{b'+1}$ — substitution law twice

$= 1=2^0 \le p \Rightarrow 2^{b'} \le p < 2^{b'+1}$ — arithmetic, given info, identity law

$= 2^{b'} \le p < 2^{b'+1}$ — which is the left side

Now we refine D :

$D \Leftarrow$ **if** $p < 2{\times}d$ **then** ok **else** $b{:=}b+1.\ d{:=}2{\times}d.\ D$ **fi**

Proof by cases. First case:

$D \Leftarrow p < 2{\times}d \wedge ok$ — expand D and ok

$= (d=2^b \le p \Rightarrow 2^{b'} \le p < 2^{b'+1}) \Leftarrow p < 2{\times}d \wedge b'=b \wedge d'=d$ — portation

$= d = 2^b \le p < 2{\times}d \wedge b'=b \wedge d'=d \Rightarrow 2^{b'} \le p < 2^{b'+1}$ — context, specialization

$\Leftarrow 2^b \le p < 2{\times}2^b \Rightarrow 2^b \le p < 2^{b+1}$ — law of arithmetic and reflexive

$= \top$

Last case:

$D \Leftarrow p \ge 2{\times}d \wedge (b{:=}b+1.\ d{:=}2{\times}d.\ D)$ — definition of D

$= (d=2^b \le p \Rightarrow 2^{b'} \le p < 2^{b'+1}) \Leftarrow p \ge 2{\times}d \wedge (b{:=}b+1.\ d{:=}2{\times}d.\ D)$ — portation

$= d=2^b \le p \wedge p \ge 2{\times}d \wedge (b{:=}b+1.\ d{:=}2{\times}d.\ D) \Rightarrow 2^{b'} \le p < 2^{b'+1}$ — definition of D

$= \quad d=2^b \le p \wedge p \ge 2{\times}d \wedge (b{:=}b+1.\ d{:=}2{\times}d.\ d=2^b \le p \Rightarrow 2^{b'} \le p < 2^{b'+1})$
$\Rightarrow 2^{b'} \le p < 2^{b'+1}$ — substitution law twice

$= \quad d=2^b \le p \wedge p \ge 2{\times}d \wedge (2{\times}d=2^{b+1} \le p \Rightarrow 2^{b'} \le p < 2^{b'+1})$
$\Rightarrow 2^{b'} \le p < 2^{b'+1}$ — arithmetic

$= \quad d=2^b \le p \wedge p \ge 2{\times}d \wedge (d=2^b \wedge p \ge 2{\times}d \Rightarrow 2^{b'} \le p < 2^{b'+1})$
$\Rightarrow 2^{b'} \le p < 2^{b'+1}$ — discharge

$= \quad d=2^b \le p \wedge p \ge 2{\times}d \wedge 2^{b'} \le p < 2^{b'+1}$
$\Rightarrow 2^{b'} \le p < 2^{b'+1}$ — specialization

$= \top$

Now the timing.

$t' \le t + log\ p \Leftarrow b{:=}0.\ d{:=}1.\ d=2^b \le p \Rightarrow t' \le t + log\ p - b$

Proof, starting with the right side:

$b{:=}0.\ d{:=}1.\ d=2^b \le p \Rightarrow t' \le t + log\ p - b$ — substitution law twice

$= 1=2^0 \le p \Rightarrow t' \le t + log\ p - 0$ — arithmetic, given $1 \le p$, identity

$= t' \le t + log\ p$ — which is the left side

Now we refine $d=2^b \le p \Rightarrow t' \le t + log\ p - b$ with recursive time:

$d=2^b \le p \Rightarrow t' \le t + log\ p - b$

$\Leftarrow$ **if** $p < 2{\times}d$ **then** ok **else** $b{:=}b+1.\ d{:=}2{\times}d.\ t{:=}t+1.\ d=2^b \le p \Rightarrow t' \le t + log\ p - b$ **fi**

Proof by cases. First case:

$(d=2^b \le p \Rightarrow t' \le t + log\ p - b) \Leftarrow p < 2{\times}d \wedge ok$ — portation and expand ok

$= d=2^b \le p < 2{\times}d \wedge b'=b \wedge d'=d \wedge t'=t \Rightarrow t' \le t + log\ p - b$ — If $2^b \le p$ then $b \le log\ p$.

$= \top$

Last case:

$(d=2^b \le p \Rightarrow t' \le t + log\ p - b)$
$\Leftarrow p \ge 2{\times}d \wedge (b{:=}b+1.\ d{:=}2{\times}d.\ t{:=}t+1.\ d=2^b \le p \Rightarrow t' \le t + log\ p - b)$
— substitution law three times

$= \quad (d=2^b \le p \Rightarrow t' \le t + log\ p - b)$
$\Leftarrow p \ge 2{\times}d \wedge (2{\times}d=2^{b+1} \le p \Rightarrow t' \le t + 1 + log\ p - (b+1))$ — arithmetic, portation

$= d=2^b \le p \wedge p \ge 2{\times}d \wedge (2{\times}d=2^{b+1} \le p \Rightarrow t' \le t + log\ p - b) \Rightarrow t' \le t + log\ p - b$
— $d=2^b$ discharges $2{\times}d=2^{b+1}$; $d=2^b$ and $p \ge 2{\times}d$ discharge $2^{b+1} \le p$

$= d=2^b \le p \wedge p \ge 2{\times}d \wedge t' \le t + log\ p - b \Rightarrow t' \le t + log\ p - b$ — specialization

$= \top$