You are given a list $L$ of at least 3 numbers such that $L_0 \geq L_1$ and $L_{(#L-2)} \leq L_{(#L-1)}$. A local minimum is an interior index $i$: 1,..,#L−1 such that $L_{(i-1)} \geq L_i \leq L_{(i+1)}$

Write a program to find a local minimum of $L$.

After trying the question, scroll down to the solution.
Specification $P$ is defined as

\[ P \equiv i': 1..#L - 1 \land L(i' - 1) \geq L \land i' \leq L(i' + 1) \]

Here is a linear search solution. Let $i$ be a natural variable.

\[ P \iff i := 1. \ Q \]

\[ Q \iff \text{if } L \land i \leq L(i + 1) \text{ then } \text{ok else } i := i + 1. \ Q \]

Now we need to define specification $Q$. Here is the first attempt: make it just like $P$ except change the 1 to $i$.

\[ Q \equiv i': 1..#L - 1 \land L(i' - 1) \geq L \land i' \leq L(i' + 1) \]

**Proof of $P$ refinement:**

\[
\begin{align*}
\text{expand } Q \\
i := 1. \ Q \\
\equiv i := 1. \ i': 1..#L - 1 \land L(i' - 1) \geq L \land i' \leq L(i' + 1) \\
\equiv i': 1..#L - 1 \land L(i' - 1) \geq L \land i' \leq L(i' + 1) \\
\equiv P
\end{align*}
\]

**Proof of $Q$ refinement, first case:**

\[
\begin{align*}
\text{expand } ok \\
L \land i \leq L(i + 1) \land \text{ok} \\
\equiv L \land i \leq L(i + 1) \land i' = i \\
\end{align*}
\]

This is not quite enough to imply $Q$. We also need $i < #L - 1$ and $L(i - 1) \geq L \land i$. So weaken $Q$.

\[
Q \equiv i < #L - 1 \land L(i - 1) \geq L \implies i': 1..#L - 1 \land L(i' - 1) \geq L \land i' \leq L(i' + 1)
\]

Now I have to redo the previous proof.

**Proof of $P$ refinement:**

\[
\begin{align*}
\text{expand } Q \\
i := 1. \ Q \\
\equiv i := 1. \ i < #L - 1 \land L(i - 1) \geq L \implies i': 1..#L - 1 \land L(i' - 1) \geq L \land i' \leq L(i' + 1) \\
\equiv 1 < #L - 1 \land L \geq L \implies i': 1..#L - 1 \land L(i' - 1) \geq L \land i' \leq L(i' + 1) \\
\equiv \top \implies i': 1..#L - 1 \land L(i' - 1) \geq L \land i' \leq L(i' + 1) \\
\equiv P
\end{align*}
\]

**Proof of $Q$ refinement, first case:**

\[
\begin{align*}
\text{expand } Q \text{ and } ok \\
Q \iff L \land i \leq L(i + 1) \land \text{ok} \\
\iff (i < #L - 1 \land L(i - 1) \geq L \implies i': 1..#L - 1 \land L(i' - 1) \geq L \land i' \leq L(i' + 1)) \\
\iff L \land i \leq L(i + 1) \land i' = i \\
\iff L \land i \leq L(i + 1) \land i' = i \land i < #L - 1 \land L(i - 1) \geq L \\
\implies i': 1..#L - 1 \land L(i' - 1) \geq L \land i' \leq L(i' + 1) \\
\equiv \top
\end{align*}
\]

**Proof of $Q$ refinement, last case:**

\[
\begin{align*}
\text{expand first } Q \text{ and portation} \\
Q \iff L \land i > L(i + 1) \land (i := i + 1. \ Q) \\
\iff i < #L - 1 \land L(i - 1) \geq L \land L \land L(i > L(i + 1) \land (i := i + 1. \ Q) \\
\implies i': 1..#L - 1 \land L(i' - 1) \geq L \land i' \leq L(i' + 1) \\
\end{align*}
\]
There may be a better solution than linear search.

So that's the plan. Now write it formally. Resuming from where I left off,

Here's my informal thinking. I see that the consequent of the inner implication

implies the main consequent

So I need to get rid of the antecedent of the inner implication. I can discharge it if I can show

which is the same as

which is the same as

The top line is \( \top \). So let's work on the bottom line.

But we are given \( L(\#L-2) \leq L(\#L-1) \). So the antecedent is \( \bot \). So the bottom line is \( \top \).

So that's the plan. Now write it formally. Resuming from where I left off,

There may be a better solution than linear search.