(Fermat’s last program) Given natural $c$, write a program to find the number of unordered pairs of naturals $a$ and $b$ such that $a^2 + b^2 = c^2$ in time proportional to $c$. (An unordered pair is really a bunch of size 1 or 2. If we have counted the pair $a$ and $b$, we don’t want to count the pair $b$ and $a$.) Your program may use addition, subtraction, multiplication, division, and comparisons, but not exponentiation or square root.

§ Define $f_{l r} = \left( \text{the number of } a \text{ and } b \text{ such that } l \leq a \leq b \leq r \land a^2 + b^2 = c^2 \right)$. Formally, $f = \langle l, r : \text{nat} \rightarrow \Sigma a, b : 0,..c+1 : \text{if } l \leq a \leq b \leq r \land a^2 + b^2 = c^2 \text{ then } 1 \text{ else } 0 \rangle$

Let’s record the answer as the final value of natural variable $n$.

\[
\begin{align*}
n' &= f_{0 c} \iff l := 0. r := c. n := 0. n' = n + f_{l r} \\
\text{else if } b > r \text{ then } ok \\
\text{else if } b \times l + r \times r > c \times c \text{ then } r := r-1. n' = n + f_{l r} \\
\text{else if } b \times l + r \times r < c \times c \text{ then } l := l+1. n' = n + f_{l r} \\
\text{else } n := n+1. l := l+1. r := r-1. n' = n + f_{l r}
\end{align*}
\]

For timing, we must prove

\[
\begin{align*}
t' &\leq t+c \iff l := 0. r := c. n := 0. l \leq r \Rightarrow t' \leq t+r-l \\
\text{if } b > r \text{ then } ok \\
\text{else if } b \times l + r \times r > c \times c \text{ then } r := r-1. t := t+1. l \leq r \Rightarrow t' \leq t+r-l \\
\text{else if } b \times l + r \times r < c \times c \text{ then } l := l+1. t := t+1. l \leq r \Rightarrow t' \leq t+r-l \\
\text{else } n := n+1. l := l+1. r := r-1. t := t+1. l \leq r \Rightarrow t' \leq t+r-l
\end{align*}
\]

Instead of $c^2$ we could use any natural $q$ (even if $q$ is not a square) in time proportional to $q^{1/2}$. To do so, we need

\[
\begin{align*}
r := \text{ceil} \left( q^{1/2} \right) &\iff r := 0. r < q^{1/2}+1 \Rightarrow q^{1/2} \leq r' < q^{1/2}+1 \\
r < q^{1/2}+1 \Rightarrow q^{1/2} \leq r' < q^{1/2}+1 &\iff \\
\text{if } r \times r \geq q \text{ then ok else } r := r+1. r < q^{1/2}+1 \Rightarrow q^{1/2} \leq r' < q^{1/2}+1
\end{align*}
\]