You are given a list $L$ of at least 3 numbers such that $L \geq L(i+1)$ and $L(\#L-2) \leq L(\#L-1)$. A local minimum is an interior index $i: 1..\#L-1$ such that $L(i-1) \geq L(i) \leq L(i+1)$.

Write a program to find a local minimum of $L$.

$\S$ Specification $P$ is defined as

$P \equiv i: 1..\#L-1 \land L(i-1) \geq L(i) \leq L(i+1)$

Here is a linear search solution. Let $i$ be a natural variable.

$P \iff i := 1. \ Q$

$Q \iff \text{if } L(i) \leq L(i+1) \text{ then } \text{ok} \text{ else } i := i+1. \ Q \fi$

Now we need to define specification $Q$. Here is the first attempt: make it just like $P$ except change the 1 to $i$.

$Q \equiv i: 1..\#L-1 \land L(i-1) \geq L(i) \leq L(i+1)$

Proof of $P$ refinement:

$i := 1. \ Q$

$\equiv i := 1. \ i: 1..\#L-1 \land L(i-1) \geq L(i) \leq L(i+1)$ expansion

$\equiv i: 1..\#L-1 \land L(i-1) \geq L(i) \leq L(i+1)$ substitution law

$\equiv P$

Proof of $Q$ refinement, first case:

$L(i) \leq L(i+1) \land \text{ok}$ expansion

$\equiv L(i) \leq L(i+1) \land i = i$ portation

This is not quite enough to imply $Q$. We also need $i < \#L-1$ and $L(i-1) \geq L(i)$. So weaken $Q$.

$Q \equiv i < \#L-1 \land L(i-1) \geq L(i) \Rightarrow i: 1..\#L-1 \land L(i-1) \geq L(i) \leq L(i+1)$

Now I have to redo the previous proof.

Proof of $P$ refinement:

$i := 1. \ Q$

$\equiv i := 1. \ i < \#L-1 \land L(i-1) \geq L(i) \Rightarrow i: 1..\#L-1 \land L(i-1) \geq L(i) \leq L(i+1)$ expansion

$\equiv 1 < \#L-1 \land L(i-1) \geq L(i) \Rightarrow i: 1..\#L-1 \land L(i-1) \geq L(i) \leq L(i+1)$ substitution law

given $\#L \geq 3$ and given $L(i-1) \geq L(i)$

$\equiv \top \Rightarrow i: 1..\#L-1 \land L(i-1) \geq L(i) \leq L(i+1)$ identity

$\equiv i: 1..\#L-1 \land L(i-1) \geq L(i) \leq L(i+1)$

$\equiv P$

Proof of $Q$ refinement, first case:

$Q \iff L(i) \leq L(i+1) \land \text{ok}$ expansion

$\iff (i < \#L-1 \land L(i-1) \geq L(i) \Rightarrow i: 1..\#L-1 \land L(i-1) \geq L(i) \leq L(i+1))$

$\iff L(i) \leq L(i+1) \land i = i$ portation

$\iff L(i) \leq L(i+1) \land i = i \land i < \#L-1 \land L(i-1) \geq L(i)$

$\Rightarrow i: 1..\#L-1 \land L(i-1) \geq L(i) \leq L(i+1)$ context

$\equiv \top$

Proof of $Q$ refinement, last case:

$Q \iff L(i) > L(i+1) \land (i := i+1. \ Q)$ expansion

$\iff i < \#L-1 \land L(i-1) \geq L(i) \land L(i) > L(i+1) \land (i := i+1. \ Q)$ expansion

$\Rightarrow i: 1..\#L-1 \land L(i-1) \geq L(i) \leq L(i+1)$ remaining $Q$
There may be a better solution than linear search.

So that's the plan. Now write it formally. Resuming from where I left off,

The top line is

which is the same as

So I need to get rid of the antecedent of the inner implication. I can discharge it if I can show

implies the main consequent

Here's my informal thinking. I see that the consequent of the inner implication

implies the main consequent

So I need to get rid of the antecedent of the inner implication. I can discharge it if I can show

which is the same as

which is the same as

The top line is \( \top \). So let's work on the bottom line.

But we are given \( L(L(-2) \leq L(#L-1) \). So the antecedent is \( \bot \). So the bottom line is \( \top \).

So that's the plan. Now write it formally. Resuming from where I left off,

\[
\begin{align*}
& i < #L-1 \land L(i-1) \geq L i \land L i > L(i+1) \\
\Rightarrow & i': i,...#L-1 \land L(i'-1) \geq L i' \leq L(i'+1) & \text{substitution law} \\
& i < #L-1 \land L(i-1) \geq L i \land L i > L(i+1) \\
\Rightarrow & i': i,...#L-1 \land L(i'-1) \geq L i' \leq L(i'+1) \\
& i < #L-1 \land L(i-1) \geq L i \land L i > L(i+1) \\
\Rightarrow & i': i,...#L-1 \land L(i'-1) \geq L i' \leq L(i'+1) \\
\end{align*}
\]

There may be a better solution than linear search.