216 (natural square root) The natural square root of a natural number \( n \) is the natural number \( s \) satisfying
\[
s^2 \leq n < (s+1)^2
\]

(a) Write a program to find the natural square root of a given natural number \( n \) in \( \log n \) time.

§ The solution is a binary search for \( s \) in the interval \([0..n+1]\).
\[
s' \leq n < (s'+1)^2 \iff s := 0. \ r := n+1. \ s^2 \leq r^2 \Rightarrow s' \leq n < (s'+1)^2 \iff
\]
\[
\text{if } s+1 = r \text{ then ok else } m := \text{div}(s+r) 2.
\]
\[
\text{if } m^2 \leq n \text{ then } s := m \text{ else } r := m. \fi.
\]
\[
s^2 \leq n < r^2 \Rightarrow s' \leq n < (s'+1)^2
\]
Execution time is identical to binary search.

(b) Write a program to find the natural square root of a given natural number \( n \) in \( \log n \) time using only addition, subtraction, doubling, halving, and comparisons (no multiplication or division).

§ The problem with the program in part (a) is the occurrence of \( m^2 \). We'll have to work in the squared space to avoid the need for squaring. Our three variables will be:
\[
s \text{ is the left end of the remaining search space times the size of the remaining search space. The left end is initially } 0, \text{ so } s \text{ is initially } 0. \text{ The size of the search space is finally } 1, \text{ so } s' \text{ is the answer.}
\]
\[
b \text{ is the square of the size of the remaining search space. To cut the search space in half, we must divide } b \text{ by } 4. \text{ To repeatedly divide } b \text{ by } 4, \text{ it must be a power of } 4.
\]
\[
n \text{ is the number whose square root is sought minus the square of the left end of the search space. Initially the left end is } 0, \text{ so } n \text{ starts as the number whose square root is sought. } n \text{ measures, in squared space, the distance from } s \text{ to the answer.}
\]
Define
\[
P \equiv \begin{align*}
n' &= n < b' : b \times 4^\text{nat} & \&(\text{if } b \leq 4 \times n \text{ then } b' \leq 4 \times n \text{ else } b' = b) \text{ fi} \land \\
t &= t + (\log(b'/b))/2
\end{align*}
\]
\[
Q \equiv \begin{align*}
n &< b + 2 \times s & (n \leq 2 \times s \Rightarrow s = 0) & (\exists m, l : b = 4^m & s = l \times 2^m) \\
\Rightarrow & b' = 1 & n' = n - s^2 \leq 2 \times s' & t' = t + (\log b)/2
\end{align*}
\]
Then
\[
s' \leq n < (s'+1)^2 \land t' \leq t + \log (n+1) + 2 \iff s := 0. \ b := 1. \ P \ Q 
\]
\[
P \iff \begin{align*}
\text{if } b > n \text{ then ok else } b := 2 \times b. \ b := 2 \times b. \ t := t+1. \ P \text{ fi}
\end{align*}
\]
\[
Q \iff \begin{align*}
\text{if } b = 1 \text{ then ok else } b := b/2. \ b := b/2. \ c := s + b. \ s := s/2. \\
\text{if } c > n \text{ then ok else } (n := n - c. \ s := s + b) \text{ fi.}
\end{align*}
\]
\[
t := t+1. \ Q \text{ fi}
\]