211 (reverse) Write a program to reverse the order of the items of a list.

After trying the question, scroll down to the solution.

§ Let L be a list variable, let k be a natural variable, and let t be time. The problem is P, defined as

$$P \;=\; \#L'{=}\#L \;\land\; (\forall n\colon 0,..\#L\cdot\ L'n{=}L(\#L{-}n{-}1)) \;\land\; t' = t + div\ (\#L)\ 2$$

Define $Q \;=\; 0{\le}k{\le}\#L/2 \;\Rightarrow\; \#L'{=}\#L \;\land\; (\forall n\colon (0,..k)\ ,(\#L{-}k,..\#L)\cdot\ L'n{=}L(\#L{-}n{-}1))$
$\qquad\qquad\qquad\qquad \land\; (\forall n\colon k,..\#L{-}k\cdot\ L'n{=}L\ n) \;\land\; t' = t{+}k$

The refinements are:

$P \;\Leftarrow\; k{:=}\ div\ (\#L)\ 2.\ \ Q$
$Q \;\Leftarrow\;$ **if** $k{=}0$ **then** ok
$\qquad\quad$ **else** $k{:=}\ k{-}1.\ \ L{:=}\ k{\to}L(\#L{-}k{-}1) \mid \#L{-}k{-}1{\to}L\ k \mid L.\ \ t{:=}\ t{+}1.\ \ Q$ **fi**

Proof:
The P refinement, starting with its right side:

$k{:=}\ div\ (\#L)\ 2.\ \ Q$ — replace Q and substitution law
$=\quad 0 \le div\ (\#L)\ 2 \le \#L/2$
$\quad\Rightarrow\quad \#L'{=}\#L$
$\qquad \land\ (\forall n\colon (0,..div\ (\#L)\ 2)\ ,(\#L{-}div\ (\#L)\ 2,..\#L)\cdot\ L'n{=}L(\#L{-}n{-}1))$
$\qquad \land\ (\forall n\colon div\ (\#L)\ 2,..\#L{-}div\ (\#L)\ 2\cdot\ L'n = L\ n)$
$\qquad \land\ \ t' = t{+}div\ (\#L)\ 2$ — case creation
$=\quad$ **if** $even\ (\#L)$
then $\ 0 \le \#L/2 \le \#L/2$
$\qquad\Rightarrow\quad \#L'{=}\#L$
$\qquad\quad \land\ (\forall n\colon (0,..\#L/2)\ ,(\#L{-}\#L/2,..\#L)\cdot\ L'n{=}L(\#L{-}n{-}1))$
$\qquad\quad \land\ (\forall n\colon \#L/2,..\#L{-}\#L/2\cdot\ L'n = L\ n)$
$\qquad\quad \land\ \ t' = t{+}div\ (\#L)\ 2$
else $\ 0 \le \#L/2 \le \#L/2$
$\qquad\Rightarrow\quad \#L'{=}\#L$
$\qquad\quad \land\ (\forall n\colon (0,..\#L/2)\ ,(\#L{-}\#L/2,..\#L)\cdot\ L'n{=}L(\#L{-}n{-}1))$
$\qquad\quad \land\ (\forall n\colon \#L/2,..\#L{-}\#L/2\cdot\ L'n = L\ n)$
$\qquad\quad \land\ \ t' = t{+}div\ (\#L)\ 2$ **fi**
$=\quad$ **if** $even\ (\#L)$
then $\ \top$
$\qquad\Rightarrow\quad \#L'{=}\#L$
$\qquad\quad \land\ (\forall n\colon 0,..\#L\cdot\ L'n{=}L(\#L{-}n{-}1))$
$\qquad\quad \land\ (\forall n\colon null\cdot\ L'n = L\ n)$
$\qquad\quad \land\ \ t' = t{+}div\ (\#L)\ 2$
else $\ \top$
$\qquad\Rightarrow\ \#L'{=}\#$
$\qquad\quad \land\ (\forall n\colon (0,..(\#L{-}1)/2), ((\#L{+}1)/2,..\#L)\cdot\ L'n{=}L(\#L{-}n{-}1))$
$\qquad\quad \land\ (\forall n\colon (\#L{-}1)/2,..\#L{-}(\#L{-}1)/2\cdot\ L'n = L\ n)$
$\qquad\quad \land\ \ t' = t{+}div\ (\#L)\ 2$ **fi**
$=\quad$ **if** $even\ (\#L)$
then $\ \#L'{=}\#L$
$\qquad \land\ (\forall n\colon 0,..\#L\cdot\ L'n{=}L(\#L{-}n{-}1))$
$\qquad \land\ \top$
$\qquad \land\ \ t' = t{+}div\ (\#L)\ 2$
else $\ \#L'{=}\#L$
$\qquad \land\ (\forall n\colon (0,..(\#L{-}1)/2), ((\#L{+}1)/2,..\#L)\cdot\ L'n{=}L(\#L{-}n{-}1))$
$\qquad \land\ (\forall n\colon (\#L{-}1)/2\cdot\ L'n = L\ n)$
$\qquad \land\ \ t' = t{+}div\ (\#L)\ 2$ **fi**

$=$ **if** $even\ (\#L)$

then $\#L'=\#L$

$\land\ (\forall n: 0,..\#L\cdot\ L'n=L(\#L-n-1))$

$\land\ \top$

$\land\ \ t' = t+div\ (\#L)\ 2$

else $\#L'=\#L$

$\land\ (\forall n: (0,..(\#L-1)/2), ((\#L+1)/2,..\#L)\cdot\ L'n=L(\#L-n-1))$

$\land\ L'((\#L-1)/2)=L(\#L-(\#L-1)/2-1)$

$\land\ \ t' = t+div\ (\#L)\ 2$ **fi**

$=$ **if** $even\ (\#L)$

then $\#L'=\#L\ \land\ (\forall n: 0,..\#L\cdot\ L'n=L(\#L-n-1))\ \land\ t' = t+div\ (\#L)\ 2$

else $\#L'=\#L\ \land\ (\forall n: 0,..\#L\cdot\ L'n=L(\#L-n-1))\ \land\ t' = t+div\ (\#L)\ 2$ **fi**

$=$ **if** $even\ (\#L)$ **then** P **else** P **fi**

$=$ P

Now the Q refinement by cases. First case:

$Q\ \Leftarrow\ k=0 \land ok$ — replace Q and ok

$=$ $(0\le k\le \#L/2\ \Rightarrow\ \#L'=\#L\ \land (\forall n: (0,..k)\ , (\#L-k,..\#L)\cdot\ L'n=L(\#L-n-1))$
$\land\ (\forall n: k,..\#L-k\cdot\ L'n=Ln)\ \land\ t' = t+k)$

$\Leftarrow\ k=0 \land L'=L \land k'=k \land t'=t$ — context

$=$ $(0\le 0\le \#L/2\ \Rightarrow\ \#L=\#L\ \land (\forall n: (0,..0)\ , (\#L,..\#L)\cdot\ L\,n = L(\#L-n-1))$
$\land\ (\forall n: 0,..\#L\cdot\ L\,n = L\,n)\ \land\ t=t)$

$\Leftarrow\ k=0 \land L'=L \land k'=k \land t'=t$ — identity and *null* domain

$=$ $(\top \Rightarrow \top \land \top \land \top \land \top)\ \Leftarrow\ k=0 \land L'=L \land k'=k \land t'=t$ — base

$=$ $\top$

Last case, right side:

$k>0 \land (k{:}= k-1.\ \ L{:}= k\rightarrow L(\#L-k-1)\ |\ \#L-k-1\rightarrow Lk\ |\ L.\ \ t{:}= t+1.\ \ Q)$ — replace Q and substitution law 3 times noting that the assignment to L does not change its length

$=$ $k>0$

$\land\ (\ \ 0 \le k-1 \le \#L/2$

$\Rightarrow\ \#L'=\#L$

$\land\ (\forall n: (0,..k-1)\ , (\#L-k+1,..\#L)\cdot$
$L'n=(k-1\rightarrow L(\#L-k+1-1)\ |\ \#L-k+1-1\rightarrow L(k-1)\ |\ L)(\#L-n-1))$

$\land\ (\forall n: k-1,..\#L-k+1\cdot\ L'n=(k-1\rightarrow L(\#L-k+1-1)\ |\ \#L-k+1-1\rightarrow L(k-1)\ |\ L)n)$

$\land\ t'=t+1+k-1)$

$=$ $k>0$

$\land\ (\ \ 0 \le k-1 \le \#L/2$

$\Rightarrow\ \#L'=\#L$

$\land\ (\forall n: (0,..k-1)\ , (\#L-k+1,..\#L)\cdot$ *

$L'n=(k-1\rightarrow L(\#L-k)\ |\ \#L-k\rightarrow L(k-1)\ |\ L)(\#L-n-1))$ **

$\land\ (\forall n: k-1,..\#L-k+1\cdot\ L'n=(k-1\rightarrow L(\#L-k)\ |\ \#L-k\rightarrow L(k-1)\ |\ L)n)$ ***

$\land\ t'=t+k)$

In the line marked ***, when $n = \#L-k$, then $L'n = L(\#L-k) = L\,n$. In the line marked **, suppose $n = \#L-k$. Then $\#L-n-1 = k-1$ and $L'n = L(\#L-k) = L\,n$. So we can move the domain element $\#L-k$ from *** to *.

In the line marked ***, when $n = k-1$, then $L'n = L(\#L-k) = L(\#L-n+1)$. In the line marked **, suppose $n = k-1$. And from context, $k-1 \le \#L/2$. Then $k-1 \neq \#L-n-1 = \#L-k$ and $L'n = L(k-1) = L\,n$. So we can move the domain element $k-1$ from *** to *.

$$
\begin{aligned}
= \quad & k>0 \\
& \wedge (\quad 0 \le k-1 \le \#L/2 \\
& \quad \Rightarrow \quad \#L'=\#L \\
& \qquad \wedge \ (\forall n: (0,..k) \ , (\#L-k,..\#L)\cdot & + \\
& \qquad\qquad L'n=(k-1\rightarrow L(\#L-k) \mid \#L-k\rightarrow L(k-1) \mid L)(\#L-n-1)) & ++ \\
& \qquad \wedge \ (\forall n: k,..\#L-k\cdot L'n=(k-1\rightarrow L(\#L-k) \mid \#L-k\rightarrow L(k-1) \mid L)n) & +++ \\
& \qquad \wedge \ t'=t+k)
\end{aligned}
$$

In the line marked ++, when $n = k-1$, then $\#L-k = \#L-n-1$. And when $n = \#L-k$, then $k-1 = \#L-n-1$. So ++ simplifies to $L'n = L(\#L-n-1)$.

In the line marked +++, n cannot be $k-1$ and n cannot be $\#L-k$, so $L'n = L\ n$.

$$
\begin{aligned}
= \quad & k>0 \\
& \wedge (\quad 0\le k\le \#L/2 \\
& \quad \Rightarrow \quad \#L'=\#L \\
& \qquad \wedge \ (\forall n: (0,..k) \ , (\#L-k,..\#L)\cdot L'n = L(\#L-n-1)) \\
& \qquad \wedge \ (\forall n: k,..\#L-k\cdot L'n = L\ n) \\
& \qquad \wedge \ t' = t+k) & \text{specialize} \\
\Rightarrow \quad & Q
\end{aligned}
$$

I made two mistakes while doing this exercise. First, I forgot the antecedent $0\le k\le \#L/2$ in Q , but I found I needed it to do the last case of the Q refinement. And I had the program wrong; I had the assignments to k and L reversed. The proof seems long and hard for such a simple program. But it made me find my program error.