206 (convex equal pair) A list of numbers is convex if its length is at least 2 , and every item (except the first and last) is less than or equal to the average of its two neighbors. Given a convex list, write a program to determine if it has a pair of consecutive equal items. Execution should be logarithmic in the length of the list.

After trying the question, scroll down to the solution.

§ Logarithmic execution time is a big clue that we should try a binary search. Look in the middle of the list; if there's an equal pair there, great; if not, eliminate half the list and look in the other half. Or, modeling the solution after binary search, use the pair in the middle of the list to eliminate half the list without checking to see if it's an equal pair.

Let the list be L. Suppose, for some m: 2,..#L, that $L(m-1) > L\,m$. Because L is convex, the first line below is a theorem, and therefore so is the last line.

$$
\begin{array}{ll}
 & L(m-1) \le (L(m-2)+Lm)/2 \qquad \text{increase } L\,m \text{ to } L(m-1) \\
\Rightarrow & L(m-1) < (L(m-2)+L(m-1))/2 \\
= & L(m-2) > L(m-1)
\end{array}
$$

By the same reasoning, all previous pairs are decreasing, and therefore unequal. Similarly if $L(m-1) < L\,m$ all following pairs are unequal. The question does not ask where a consecutive pair of equal items is, but only whether there is such a pair. So my specification is R, where

$$R \;=\; p' = (\exists i\colon 1,..\#L\cdot L(i-1) = L\,i)$$

Introduce indexes h and j and define

$$Q \;=\; 0<h<j\le\#L \;\Rightarrow\; p' = (\exists i\colon h,..j\cdot L(i-1) = L\,i)$$

We now solve the problem.

$R \;\Leftarrow\; h:= 1.\; j:= \#L.\; Q$

$Q \;\Leftarrow\;$ **if** $j-h = 1$ **then** $p:= L(h-1) = L\,h$
else $m:= div\;(h+j)\;2.$
if $L(m-1) > L\,m$ **then** $h:= m$ **else** $j:= m$ **fi**.
Q **fi**

Here is the proof of the first refinement, starting with the right side.

$$
\begin{array}{ll}
 & h:= 1.\; j:= \#L.\; Q \qquad \text{substitution law twice} \\
= & 0<1<\#L\le\#L \;\Rightarrow\; p' = (\exists i\colon 1,..\#L\cdot L(i-1) = L\,i) \qquad L \text{ is convex therefore } 1<\#L \\
= & R
\end{array}
$$

For the second refinement, the main **else** becomes

$m:= div\;(h+j)\;2.$
if $L(m-1) > L\,m$ **then** $h:= m$ **else** $j:= m$ **fi**.
Q — distributivity and substitution law

= **if** $m:= div\;(h+j)\;2.\; L(m-1) > L\,m$ **then** $m:= div\;(h+j)\;2.\; h:= m.\; Q$ — substitution
else $m:= div\;(h+j)\;2.\; j:= m.\; Q$ **fi**

= **if** $L((div\;(h+j)\;2)-1) > L(div\;(h+j)\;2)$ **then** $m:= div\;(h+j)\;2.\; h:= m.\; Q$
else $m:= div\;(h+j)\;2.\; j:= m.\; Q$ **fi**

So the refinement

$Q \;\Leftarrow\;$ **if** $j-h = 1$ **then** $p:= L(h-1) = L\,h$
else if $L((div\;(h+j)\;2)-1)>L(div\;(h+j)\;2)$ **then** $m:= div(h+j)2.\; h:= m.\; Q$
else $m:= div\;(h+j)\;2.\; j:= m.\; Q$ **fi fi**

can now be treated in three cases. First case:

$$
\begin{array}{ll}
 & Q \;\Leftarrow\; j-h = 1 \wedge (p:= L(h-1) = L\,h) \qquad \text{replace } Q \\
= & (0<h<j\le\#L \;\Rightarrow\; p' = (\exists i\colon h,..j\cdot L(i-1) = L\,i)) \;\Leftarrow\; j-h = 1 \wedge (p:= L(h-1) = L\,h)
\end{array}
$$

portation

$$
\begin{array}{ll}
= & 0<h<j\le\#L \wedge j-h=1 \wedge (p:= L(h-1) = L\,h) \Rightarrow p' = (\exists i\colon h,..j\cdot L(i-1) = L\,i) \qquad \text{assignment} \\
= & \quad 0<h<j\le\#L \wedge j-h=1 \wedge p'=(L(h-1) = L\,h) \wedge m'=m \wedge h'=h \wedge j'=j \\
 & \Rightarrow p' = (\exists i\colon h,..j\cdot L(i-1) = L\,i) \qquad \text{generalization, context} \\
\Leftarrow & \top
\end{array}
$$

Middle case:

$$
\begin{array}{ll}
 & Q \;\Leftarrow\; j-h \ne 1 \wedge L((div\;(h+j)\;2)-1)>L(div\;(h+j)\;2) \wedge (m:= div\;(h+j)\;2.\; h:= m.\; Q) \\
 & \qquad\qquad \text{replace } Q \\
= & \quad (0<h<j\le\#L \;\Rightarrow\; p' = (\exists i\colon h,..j\cdot L(i-1) = L\,i)) \\
 \Leftarrow & \quad j-h \ne 1 \wedge L((div\;(h+j)\;2)-1)>L(div\;(h+j)\;2) \\
 & \quad \wedge\, (m:= div\;(h+j)\;2.\; h:= m.\; (0<h<j\le\#L \;\Rightarrow\; p' = (\exists i\colon h,..j\cdot L(i-1) = L\,i)))
\end{array}
$$

portation

$=$ $0<h<j\le\#L \wedge j-h\ne 1 \wedge L((div\ (h+j)\ 2)-1)>L(div\ (h+j)\ 2)$
$\wedge\ (m:=div\ (h+j)\ 2.\ \ h:=m.\ \ (0<h<j\le\#L \Rightarrow p' = (\exists i\colon h,..j\cdot L(i-1)=L\ i)))$
$\Rightarrow\ p' = (\exists i\colon h,..j\cdot L(i-1)=L\ i)$ substitution law twice

$=$ $0<h<j\le\#L \wedge j-h\ne 1 \wedge L((div\ (h+j)\ 2)-1)>L(div\ (h+j)\ 2)$
$\wedge\ (0 < div\ (h+j)\ 2 < j \le \#L \Rightarrow p' = (\exists i\colon div\ (h+j)\ 2\ ,..\ j\cdot L(i-1)=L\ i))$
$\Rightarrow\ p' = (\exists i\colon h,..j\cdot L(i-1)=L\ i)$ To use the Law of Discharge we must prove that the main antecedent implies the inner antecedent; so we will prove $0<h<j\le\#L \Rightarrow 0 < div\ (h+j)\ 2 < j \le \#L$ later; for now, assume it, and use the law of Discharge

$=$ $0<h<j\le\#L \wedge j-h\ne 1 \wedge L((div\ (h+j)\ 2)-1)>L(div\ (h+j)\ 2)$
$\wedge\ p' = (\exists i\colon div\ (h+j)\ 2\ ,..\ j\cdot L(i-1)=L\ i))$
$\Rightarrow\ p' = (\exists i\colon h,..j\cdot L(i-1)=Li)$ Now we use the earlier observation that if $L((div\ (h+j)\ 2)-1)>L(div\ (h+j)\ 2)$ then all items with indexes less than $div\ (h+j)\ 2$ are unequal, and so $(\exists i\colon div\ (h+j)\ 2\ ,..\ j\cdot L(i-1)=L\ i) = (\exists i\colon h,..j\cdot L(i-1)=L\ i)$.

$\Leftarrow$ $\top$

As promised, we prove $0<h<j\le\#L \Rightarrow 0 < div\ (h+j)\ 2 < j \le \#L$.

$0<h<j\le\#L$

$=$ $1\le h \wedge 2\le j \wedge h<j \wedge j\le\#L$

$\Rightarrow$ $3 \le h+j < j+j \wedge j\le\#L$

$\Rightarrow$ $0 < div\ (h+j)\ 2 < j \le \#L$

Last case:

$Q \Leftarrow j-h \ne 1 \wedge L((div\ (h+j)\ 2)-1)\le L(div\ (h+j)\ 2) \wedge (m:=div\ (h+j)\ 2.\ \ j:=m.\ \ Q)$
as in the middle case, replace Q, portation, substitution law twice

$=$ $0<h<j\le\#L \wedge j-h\ne 1 \wedge L((div\ (h+j)\ 2)-1)\le L(div\ (h+j)\ 2)$
$\wedge\ (0 < h < div\ (h+j)\ 2 \le \#L \Rightarrow p' = (\exists i\colon h\ ,..\ div\ (h+j)\ 2\cdot L(i-1)=L\ i))$
$\Rightarrow\ p' = (\exists i\colon h,..j\cdot L(i-1)=L\ i)$ To use the Law of Discharge we must prove that the main antecedent implies the inner antecedent; so we will prove $0<h<j\le\#L \wedge j-h\ne 1 \Rightarrow 0 < h < div\ (h+j)\ 2 \le \#L$ later; for now, assume it, and use the law of Discharge

$=$ $0<h<j\le\#L \wedge j-h\ne 1 \wedge L((div\ (h+j)\ 2)-1)\le L(div\ (h+j)\ 2)$
$\wedge\ p' = (\exists i\colon h\ ,..\ div\ (h+j)\ 2\cdot L(i-1)=L\ i))$
$\Rightarrow\ p' = (\exists i\colon h,..j\cdot L(i-1)=L\ i)$ The antecedent says $L((div\ (h+j)\ 2)-1)\le L(div\ (h+j)\ 2)$. If $L((div\ (h+j)\ 2)-1)=L(div\ (h+j)\ 2)$ then both $\exists i\colon h\ ,..\ div\ (h+j)\ 2\cdot L(i-1)=L\ i)$ and $\exists i\colon h,..j\cdot L(i-1)=L\ i$ are $\top$. If $L((div\ (h+j)\ 2)-1)<L(div\ (h+j)\ 2)$ we use the earlier observation that all items with indexes more than $div\ (h+j)\ 2$ are unequal, and so $(\exists i\colon div\ (h+j)\ 2,..\ j\cdot L(i-1)=L\ i) = (\exists i\colon h,..j\cdot L(i-1)=L\ i)$.

$\Leftarrow$ $\top$

As promised, we prove $0<h<j\le\#L \wedge j-h\ne 1 \Rightarrow 0 < h < div\ (h+j)\ 2 \le \#L$.

$0<h<j\le\#L \wedge j-h\ne 1 \Rightarrow 0 < h < div\ (h+j)\ 2 \le \#L$

$\Leftarrow$ $(0<h \Rightarrow 0<h) \wedge (h<j \wedge j-h\ne 1 \Rightarrow h < div\ (h+j)\ 2) \wedge (h<j\le\#L \Rightarrow div\ (h+j)\ 2 \le \#L)$

$\Leftarrow$ $\top$

In both the middle case and final cases, there are places where the hint is too large and too chatty, and I would like to improve the proof.
The timing analysis is identical to binary search.