205 (ordered pair search) Given a list of at least two items whose first item is less than or equal to its last item, write a program to find an adjacent pair of items such that the first of the pair is less than or equal to the second of the pair. Execution time should be logarithmic in the length of the list.

After trying the question, scroll down to the solution.

§ This one is a binary search. Let the list be L, and let i, j, and m be natural variables.

$$L\,i' \le L(i'+1) \;\Longleftarrow\; i{:=}\,0.\;\; j{:=}\,\#L-1.\;\; 0{\le}i{<}j{<}\#L \land L\,i \le L\,j \;\Rightarrow\; L\,i' \le L(i'+1)$$

$$0{\le}i{<}j{<}\#L \;\land\; L\,i \le L\,j \;\Rightarrow\; L\,i' \le L(i'+1) \;\Longleftarrow$$
$$\begin{array}{l}\textbf{if } i{+}1{=}j \textbf{ then } ok\\ \textbf{else}\;\; m{:=}\,div\,(i{+}j)\,2.\\ \qquad \textbf{if } L\,i \le L\,m \textbf{ then } j{:=}\,m \textbf{ else } i{:=}\,m \textbf{ fi}.\\ \qquad 0{\le}i{<}j{<}\#L \;\land\; L\,i \le L\,j \;\Rightarrow\; L\,i' \le L(i'+1) \textbf{ fi}\end{array}$$

The first of these two refinements is trivial to prove. The second will be proven by cases. First case:

$$(0{\le}i{<}j{<}\#L \land L\,i \le L\,j \;\Rightarrow\; L\,i' \le L(i'+1) \;\Longleftarrow\; i{+}1{=}j \land ok) \qquad \text{portation and expand } ok$$
$$= \; 0{\le}i{<}j{<}\#L \land L\,i \le L\,j \;\land i{+}1{=}j \land i'{=}i \land j'{=}j \;\Rightarrow\; L\,i' \le L(i'+1)$$
$$= \; \top$$

Second case:

$$\begin{array}{ll}(& 0{\le}i{<}j{<}\#L \land L\,i \le L\,j \;\Rightarrow\; L\,i' \le L(i'+1)\\ \Longleftarrow & i{+}1{\ne}j \land (\; m{:=}\,div\,(i{+}j)\,2.\\ & \qquad L\,i \le L\,m \land j{:=}\,m.\\ & \qquad 0{\le}i{<}j{<}\#L \land L\,i \le L\,j \;\Rightarrow\; L\,i' \le L(i'+1)\;)\;)\end{array}$$

portation, substitutions, condition law

$$\begin{array}{ll} = & \quad 0{\le}i{<}j{<}\#L \land L\,i \le L\,j \;\land i{+}1{\ne}j \land L\,i \le L(div\,(i{+}j)\,2)\\ & \land\;\; (0{\le}i{<}div\,(i{+}j)\,2{<}\#L \land L\,i \le L(div\,(i{+}j)\,2) \;\Rightarrow\; L\,i' \le L(i'+1))\\ & \Rightarrow\; L\,i' \le L(i'+1) \end{array}$$

which requires discharge

$$\begin{array}{ll} \Longleftarrow & 0{\le}i{<}j{<}\#L \land L\,i \le L\,j \;\land i{+}1{\ne}j \land L\,i \le L(div\,(i{+}j)\,2)\\ & \Rightarrow\; 0{\le}i{<}div\,(i{+}j)\,2{<}\#L \land L\,i \le L(div\,(i{+}j)\,2)\\ = & \top \end{array}$$

Third case:

$$\begin{array}{ll}(& 0{\le}i{<}j{<}\#L \land L\,i \le L\,j \;\Rightarrow\; L\,i' \le L(i'+1)\\ \Longleftarrow & i{+}1{\ne}j \land (\; m{:=}\,div\,(i{+}j)\,2.\\ & \qquad L\,i < L\,m \land i{:=}\,m.\\ & \qquad 0{\le}i{<}j{<}\#L \land L\,i \le L\,j \;\Rightarrow\; L\,i' \le L(i'+1)\;)\;)\end{array}$$

just like the second case.

$$= \; \top$$

That's the end of the proof, but we still have to look at the time.

$$t' \le t + log(\#L-1) \;\Longleftarrow\; i{:=}\,0.\;\; j{:=}\,\#L-1.\;\; i{<}j \;\Rightarrow\; t' \le t + log(j-i)$$

$$i{<}j \;\Rightarrow\; t' \le t + log(j-i) \;\Longleftarrow$$
$$\begin{array}{l}\textbf{if } i{+}1{=}j \textbf{ then } ok\\ \textbf{else}\;\; m{:=}\,div\,(i{+}j)\,2.\\ \qquad \textbf{if } L\,i \le L\,m \textbf{ then } j{:=}\,m \textbf{ else } i{:=}\,m \textbf{ fi}.\\ \qquad t{:=}\,t{+}1.\;\; i{<}j \;\Rightarrow\; t' \le t + log(j-i) \textbf{ fi}\end{array}$$

The first refinement is proven by two substitutions. The second is proven by cases. First,

$$(i{<}j \;\Rightarrow\; t' \le t + log(j-i) \;\Longleftarrow\; i{+}1{=}j \land ok)$$
$$= \; i{+}1{=}j \land i'{=}i \land j'{=}j \land t'{=}t \land i{<}j \;\Rightarrow\; t' \le t + log(j-i) \qquad \text{use } log\;1 = 0$$
$$= \; \top$$

Second,

$$\begin{array}{ll}(& i{<}j \;\Rightarrow\; t' \le t + log(j-i)\\ \Longleftarrow & i{+}1{\ne}j \land (\; m{:=}\,div\,(i{+}j)\,2.\\ & \qquad L\,i \le L\,m \land j{:=}\,m.\\ & \qquad t{:=}\,t{+}1.\;\; i{<}j \;\Rightarrow\; t' \le t + log(j-i)\;)\;)\end{array}$$

portation, substitutions, condition law

$$\begin{array}{ll} = & \quad i{<}j \land i{+}1{\ne}j \land L\,i \le L(div\,(i{+}j)\,2)\\ & \land\;\; (i{<}div\,(i{+}j)\,2 \;\Rightarrow\; t' \le t{+}1{+}log(div\,(i{+}j)\,2 - i))\\ & \Rightarrow\;\; t' \le t + log(j-i) \end{array}$$

use $i{<}j \land i{+}1{\ne}j$ to discharge $i{<}div\,(i{+}j)\,2$. Drop useless conjunct in antecedent.

$\Leftarrow\quad i<j \wedge i+1 \neq j \wedge t' \le t+1+log(div\ (i+j)\ 2 - i) \ \Rightarrow\ t' \le t + log(j-i)$

$=\quad i<j \wedge i+1 \neq j \ \Rightarrow\ 1+log(div\ (i+j)\ 2 - i) \le log(j-i)$

$\Leftarrow\quad i<j \wedge i+1 \neq j \ \Rightarrow\ 1+log((i+j)/2 - i) \le log(j-i)$

$=\quad i<j \wedge i+1 \neq j \ \Rightarrow\ log(j-i) \le log(j-i)$

$=\quad \top$

Third case:

$$
\begin{aligned}
&(\quad i<j \ \Rightarrow\ t' \le t + log(j-i) \\
&\Leftarrow\quad i+1 \neq j \wedge (\ m := div\ (i+j)\ 2. \\
&\qquad\qquad L\ i < L\ m \wedge i := m. \\
&\qquad\qquad t := t+1.\ \ i<j \ \Rightarrow\ t' \le t + log(j-i)\ )\ ) \qquad \text{just like the second case} \\
&= \quad \top
\end{aligned}
$$