195 (fixed point) Let L be a nonempty sorted list of different integers. Write a program to find a fixed-point of L, that is an index i such that $L\ i = i$, or to report that no such index exists. Execution time should be at most $log\ (\#L)$.

After trying the question, scroll down to the solution.

§ Let L be a constant, and let i and j be a natural variables. Let t be an extended natural time variable. If a fixed-point exists, it will be indicated by $L\,i' = i'$. If none exists, that will be indicated by $L\,i' \ne i'$.

$$\begin{array}{l}
(\exists k\colon 0,..\#L\cdot L\,k = k) = (L\,i' = i') \;\Leftarrow \\
\qquad i{:=}\,0.\;\; j{:=}\,\#L.\;\; i{<}j \;\Rightarrow\; (\exists k\colon i,..j\cdot L\,k = k) = (L\,i' = i')
\end{array}$$

$$\begin{array}{l}
i{<}j \;\Rightarrow\; (\exists k\colon i,..j\cdot L\,k = k) = (L\,i' = i') \;\Leftarrow \\
\qquad \textbf{if } j{-}i{=}1 \textbf{ then } ok \\
\qquad \textbf{else}\;\; m{:=}\, div\,(i{+}j)\;2. \\
\qquad\qquad \textbf{if } L\,m \le m \textbf{ then } i{:=}\,m \textbf{ else } j{:=}\,m \textbf{ fi}. \\
\qquad\qquad i{<}j \;\Rightarrow\; (\exists k\colon i,..j\cdot L\,k = k) = (L\,i' = i') \textbf{ fi}
\end{array}$$

The timing:

$$t' \le t + ceil\,(log\,(\#L)) \;\Leftarrow\; i{:=}\,0.\;\; j{:=}\,\#L.\;\; i{<}j \Rightarrow t' \le t + ceil\,(log\,(j{-}i))$$

$$\begin{array}{l}
i{<}j \Rightarrow t' \le t + ceil\,(log\,(j{-}i)) \;\Leftarrow \\
\qquad \textbf{if } j{-}i{=}1 \textbf{ then } ok \\
\qquad \textbf{else}\;\; m{:=}\, div\,(i{+}j)\;2. \\
\qquad\qquad \textbf{if } L\,m \le m \textbf{ then } i{:=}\,m \textbf{ else } j{:=}\,m \textbf{ fi}. \\
\qquad\qquad t{:=}\,t{+}1.\;\; i{<}j \Rightarrow t' \le t + ceil\,(log\,(j{-}i)) \textbf{ fi}
\end{array}$$

The first refinement is proven by two uses of the Substitution Law. The last refinement is proven in three cases. First case:

$$\begin{array}{lll}
 & (i{<}j \;\Rightarrow\; (\exists k\colon i,..j\cdot L\,k = k) = (L\,i' = i') \;\Leftarrow\; j{-}i{=}1 \wedge ok) & \text{expand } ok \\
= & (i{<}j \;\Rightarrow\; (\exists k\colon i,..j\cdot L\,k = k) = (L\,i' = i') \;\Leftarrow\; j{-}i{=}1 \wedge i'{=}i \wedge j'{=}j) & \text{context} \\
= & (i{<}i{+}1 \;\Rightarrow\; (\exists k\colon i,..i{+}1\cdot L\,k = k) = (L\,i = i) \;\Leftarrow\; j{-}i{=}1 \wedge i'{=}i \wedge j'{=}j) & \text{simplify} \\
= & (\top \;\Rightarrow\; (L\,i = i) = (L\,i = i) \;\Leftarrow\; j{-}i{=}1 \wedge i'{=}i \wedge j'{=}j) & \text{reflexive and identity} \\
= & \top &
\end{array}$$

Middle case:

$$\begin{array}{lll}
 & (\qquad i{<}j \;\Rightarrow\; (\exists k\colon i,..j\cdot L\,k = k) = (L\,i' = i') & \\
 & \Leftarrow\; j{-}i{\ne}1 \wedge (\; m{:=}\, div\,(i{+}j)\;2. & \\
 & \qquad\qquad L\,m \le m \wedge (i{:=}\,m.\;\; i{<}j \;\Rightarrow\; (\exists k\colon i,..j\cdot L\,k = k) = (L\,i' = i')))) & \\
 & & \text{portation} \\
= & \quad j{-}i{\ge}2 \;\wedge\; (\; m{:=}\, div\,(i{+}j)\;2. & \\
 & \qquad\qquad L\,m \le m \wedge (i{:=}\,m.\;\; i{<}j \;\Rightarrow\; (\exists k\colon i,..j\cdot L\,k = k) = (L\,i' = i'))) & \\
 & \Rightarrow\; (\exists k\colon i,..j\cdot L\,k = k) = (L\,i' = i') & \text{Substitution Law twice} \\
= & \quad j{-}i{\ge}2 \;\wedge\; L(div\,(i{+}j)\;2) {\le} (div\,(i{+}j)\;2) & \\
 & \wedge\; ((div\,(i{+}j)\;2){<}j \;\Rightarrow\; (\exists k\colon (div\,(i{+}j)\;2),..j\cdot L\,k = k) = (L\,i' = i')) & \\
 & \Rightarrow\; (\exists k\colon i,..j\cdot L\,k = k) = (L\,i' = i') & \\
 & & \text{In the context } j{-}i{\ge}2 \text{, we have } (div\,(i{+}j)\;2){<}j \text{, so discharge} \\
= & \quad j{-}i{\ge}2 \;\wedge\; L(div\,(i{+}j)\;2){\le}(div\,(i{+}j)\;2) \;\wedge\; (\exists k\colon (div\,(i{+}j)\;2),..j\cdot L\,k = k) = (L\,i' = i') & \\
 & \Rightarrow\; (\exists k\colon i,..j\cdot L\,k = k) = (L\,i' = i') & \\
 & & \text{If } L(div\,(i{+}j)\;2){=}(div\,(i{+}j)\;2) \text{, then } (\exists k\colon (div\,(i{+}j)\;2),..j\cdot L\,k = k) \text{ and} \\
 & & (\exists k\colon i,..j\cdot L\,k = k) \text{ are both } \top . \\
 & & \text{If } L(div\,(i{+}j)\;2){<}(div\,(i{+}j)\;2) \text{, then } (\exists k\colon i,..(div\,(i{+}j)\;2)\cdot L\,k = k) \text{ is } \bot \\
 & & \text{because } L \text{ is strictly increasing.} \\
= & \quad j{-}i{\ge}2 \;\wedge\; L(div\,(i{+}j)\;2){\le}(div\,(i{+}j)\;2) \;\wedge\; (\exists k\colon i,..j\cdot L\,k = k) = (L\,i' = i') & \\
 & \Rightarrow\; (\exists k\colon i,..j\cdot L\,k = k) = (L\,i' = i') & \text{specialization} \\
= & \top &
\end{array}$$

The last case is just like the middle case. The timing proof breaks into the same cases as the results proof.