(sorted two-dimensional search) Write a program to find a given item in a given 2-dimensional array in which each row is sorted and each column is sorted. The execution time must be linear in the sum of the dimensions.

Let the array be $A$, let its dimensions be $n$ by $m$, and let the item we seek be $x$. The problem, except for time, is $P$, where

$$P = \text{if } x: A(0..n)(0..m) \text{ then } x = A[i']j' \text{ else } i' = -1 \vee j' = m \text{ fi}$$

The idea is to start at the lower left corner of the array, and by comparing that item with $x$ we can cross off an entire row or column, and then repeat. We'll need integer variables $i$ and $j$ to keep track of the row and column. Define

$$Q = -1 \leq i < n \land 0 \leq j < m \Rightarrow \text{if } x: A(0..i+1)(j..m) \text{ then } x = A[i']j' \text{ else } i' = -1 \vee j' = m \text{ fi}$$

which specifies the search in the clear part of the picture.

Here is the proof. First the refinement of $P$.

$$i := n - 1 \land j := 0. \quad Q \quad \text{expand } Q; \text{ substitution law twice}$$

$$\Rightarrow -1 \leq n - 1 < n \land 0 \leq 0 < m \Rightarrow \text{if } x: A(0..i+1)(j..m) \text{ then } x = A[i']j' \text{ else } i' = -1 \vee j' = m \text{ fi}$$

$$= P$$

Now the refinement of $Q$. We use case analysis.

$$Q \Leftarrow (i = -1 \lor j = m) \land ok \quad \text{expand } Q, \text{ portation}$$

$$\Rightarrow \text{if } x: A(0..i+1)(j..m) \text{ then } x = A[i']j' \text{ else } i' = -1 \lor j' = m \text{ fi} \quad \text{distribution, antidist}$$

$$= (\text{if } x: A(0..i+1)(j..m) \text{ then } x = A[i']j' \text{ else } i' = -1 \lor j' = m \text{ fi})$$

$$\land (j = m \land i = -1 \lor 0 \leq j \leq m) \land -1 \leq i < n \land 0 \leq j \leq m$$

$$\Rightarrow \text{if } x: A(0..i+1)(j..m) \text{ then } x = A[i']j' \text{ else } i' = -1 \lor j' = m \text{ fi} \quad \text{expand ok}$$

$$= (\text{if } x: A(0..i+1)(j..m) \text{ then } x = A[i']j' \text{ else } i' = -1 \lor j' = m \text{ fi})$$

$$\land (j = m \land i = -1 \lor 0 \leq j \leq m) \land -1 \leq i < n \land 0 \leq j \leq m$$

$$\Rightarrow \text{if } x: A(0..i+1)(j..m) \text{ then } x = A[i']j' \text{ else } i' = -1 \lor j' = m \text{ fi} \quad \text{antecedent context}$$

$$= (\text{if } x: A(0..i+1)(j..m) \text{ then } x = A[i']j' \text{ else } i' = -1 \lor j' = m \text{ fi})$$

$$\land (j = m \land i = -1 \lor 0 \leq j \leq m) \land -1 \leq i < n \land 0 \leq j \leq m$$

$$\Rightarrow \text{if } x: A(0..i+1)(j..m) \text{ then } x = A[i']j' \text{ else } i' = -1 \lor j' = m \text{ fi}$$

$$= T$$
The timing proof is much easier. $P$ becomes $t' \leq t+n+m$ and $Q$ becomes $-1 \leq i < n \land 0 \leq j \leq m \Rightarrow t' \leq t+i+1+m-j$