181 (multiplication table) Given n: *nat* and variable M: [*[*nat*]] , write a program to assign to M a multiplication table of size n without using multiplication. For example, if $n = 4$, then

$$M' = [\ [0];\ [0;\ 1];\ [0;\ 2;\ 4];\ [0;\ 3;\ 6;\ 9]\ ]$$

After trying the question, scroll down to the solution.

§ Extend the definition of multiplication to allow the right operand to be a string.

$i \times nil \ = \ nil$

$i \times (j ; S) \ = \ i{\times}j ; i{\times}S$

Define $row \ = \ \langle i{:}\ nat \rightarrow [i \times (0;..i+1)]\rangle$. For example,

$row\ 3 \ = \ [3{\times}(0; 1; 2; 3)] \ = \ [3{\times}0; 3{\times}1; 3{\times}2; 3{\times}3] \ = \ [0; 3; 6; 9]$

So the problem is $M' \ = \ row\ [0;..n]$. Introduce new variables $i, j{:}\ nat$ and $R{:}\ [*nat]$.

$M' = row\ [0;..n] \ \Leftarrow \ M{:=}\ [nil].\ i{:=}\ 0.\ M' = M\ ;;\ row\ [i;..n]$

$M' = M\ ;;\ row\ [i;..n] \ \Leftarrow$

if $i{=}n$ **then** *ok*

else $R' = row\ i \wedge M'{=}M \wedge i'{=}i.\ M{:=}\ M\ ;;\ [row\ i].\ i{:=}\ i+1.\ M' = M\ ;;\ row\ [i;..n]$ **fi**

$R' = row\ i \ \wedge \ M'{=}M \ \wedge \ i'{=}i \ \Leftarrow$

$R{:=}\ [nil].\ j{:=}\ 0.\ R' = R\ ;;\ [i \times (j;..i+1)] \wedge M'{=}M \wedge i'{=}i$

$R' = R\ ;;\ [i \times (j;..i+1)] \wedge M'{=}M \wedge i'{=}i \ \Leftarrow$

if $j = i+1$ **then** *ok*

else $R{:=}\ R\ ;;\ [i{\times}j].\ j{:=}\ j+1.\ R' = R\ ;;\ [i \times (j;..i+1)] \wedge M'{=}M \wedge i'{=}i$ **fi**

Recursive time requires $t{:=}\ t+1$ in the inner loop.

$t' \le t + n{\times}(n-1)/2 \ \Leftarrow \ M{:=}\ [nil].\ i{:=}\ 0.\ t' \le t + n{\times}(n-1)/2 - i{\times}(i-1)/2$

$t' \le t + n{\times}(n-1)/2 - i{\times}(i-1)/2 \ \Leftarrow$

if $i{=}n$ **then** *ok*

else $t' \le t+i+1 \wedge i'{=}i.\ M{:=}\ M\ ;;\ [row\ i].\ i{:=}\ i+1.\ t' \le t + n{\times}(n-1)/2 - i{\times}(i-1)/2$ **fi**

$t' \le t+i+1 \wedge i'{=}i \ \Leftarrow$

$R{:=}\ [nil].\ j{:=}\ 0.\ t' \le t+i+1-j \wedge i'{=}i$

$t' \le t+i+1-j \wedge i'{=}i \ \Leftarrow$

if $j = i+1$ **then** *ok*

else $R{:=}\ R\ ;;\ [i{\times}j].\ j{:=}\ j+1.\ t{:=}\ t+1.\ t' \le t+i+1-j \wedge i'{=}i$ **fi**