179√ (binary exponentiation) Given natural variables x and y, write a program for $y' = 2^x$ without using exponentiation.

§ see book Subsection 4.1.2 and scroll down for proofs. See also Subsection 5.2.3.

In the textbook on page 44 there is the solution

0 $y'=2^x \Leftarrow$ **if** $x=0$ **then** $x=0 \Rightarrow y'=2^x$ **else** $x>0 \Rightarrow y'=2^x$ **fi**
1 $x=0 \Rightarrow y'=2^x \Leftarrow y:=1.\ x:=3$
2 $x>0 \Rightarrow y'=2^x \Leftarrow x>0 \Rightarrow y'=2^{x-1}.\ y'=2\times y$
3 $x>0 \Rightarrow y'=2^{x-1} \Leftarrow x'=x-1.\ y'=2^x$
4 $y'=2\times y \Leftarrow y:=2\times y.\ x:=5$
5 $x'=x-1 \Leftarrow x:=x-1.\ y:=7$

Proof of refinement 0.

	$y'=2^x$	case creation (x is natural)
$=$	**if** $x=0$ **then** $x=0 \Rightarrow y'=2^x$ **else** $x>0 \Rightarrow y'=2^x$ **fi**	

Proof of refinement 1.

	$(x=0 \Rightarrow y'=2^x \Leftarrow y:=1.\ x:=3)$	portation
$=$	$x=0 \wedge (y:=1.\ x:=3) \Rightarrow y'=2^x$	definition of assignment and substitution law
$=$	$x=0 \wedge y'=1 \wedge x'=3 \Rightarrow y'=2^x$	context
$=$	$x=0 \wedge y'=1 \wedge x'=3 \Rightarrow 1=2^0$	arithmetic
$=$	$x=0 \wedge y'=1 \wedge x'=3 \Rightarrow \top$	base
$=$	$\top$	

Proof of refinement 2.

	$x>0 \Rightarrow y'=2^{x-1}.\ y'=2\times y$	sequential composition
$=$	$\exists x'', y''\cdot\ (x>0 \Rightarrow y''=2^{x-1}) \wedge y'=2\times y''$	idempotent (x'' does not appear)
$=$	$\exists y''\cdot\ (x>0 \Rightarrow y''=2^{x-1}) \wedge y'=2\times y''$	arithmetic
$=$	$\exists y''\cdot\ (x>0 \Rightarrow y''=2^{x-1}) \wedge y''=y'/2$	one-point
$=$	$x>0 \Rightarrow y'/2 = 2^{x-1}$	arithmetic
$=$	$x>0 \Rightarrow y'=2^x$	

Proof of refinement 3.

	$(x>0 \Rightarrow y'=2^{x-1} \Leftarrow x'=x-1.\ y'=2^x)$	portation
$=$	$x>0 \wedge (x'=x-1.\ y'=2^x) \Rightarrow y'=2^{x-1}$	sequential composition
$=$	$x>0 \wedge (\exists x'', y''\cdot\ x''=x-1 \wedge y'=2^{x''}) \Rightarrow y'=2^{x-1}$	idempotent and one-point
$=$	$x>0 \wedge y'=2^{x-1} \Rightarrow y'=2^{x-1}$	specialization
$=$	$\top$	

Proof of refinement 4.

	$y:=2\times y.\ x:=5$	definition of assignment and substitution law
$=$	$x'=5 \wedge y'=2\times y$	specialization
$\Rightarrow$	$y'=2\times y$	

Proof of refinement 5.

	$x:=x-1.\ y:=7$	definition of assignment and substitution law
$=$	$x'=x-1 \wedge y'=7$	specialization
$\Rightarrow$	$x'=x-1$	