Write a program to find the number of ways to partition $a+b$ things into $a$ things in the left part and $b$ things in the right part. Include recursive time.

After trying the question, scroll down to the solution.
The number of ways to partition \(a+b\) things into \(a\) things and \(b\) things is \((a+b)! / (a!b!)\) where \(!\) is the factorial function. First without time.

\[
x := (a+b)! / (a!b!) \iff
\]

\[
\text{if } a=0 \text{ then } x := 1
\]
\[
\text{else } a := a-1. \ x := (a+b)! / (a!b!). \ a := a+1. \ x := x \times (a+b)/a \ \
\]

The assignment \(x := (a+b)! / (a!b!)\) means \(x' := (a+b)! / (a!b!)) \wedge a' = a \wedge b' = b\). On the right side it is a recursive call. Stating it as an assignment makes the proof easy: just use the substitution law and simplify. The proof is by cases. First case:

\[
a = 0 \land (x := 1) \Rightarrow (x := (a+b)! / (a!b!)) \quad \text{definition of assignment}
\]

\[
= a = 0 \land x' = 1 \land a' = a \land b' = b \Rightarrow x' := (a+b)! / (a!b!) \land a' = a \land b' = b \quad \text{use } 0! = 1
\]

\[
= T
\]

Second case, starting with the right side:

\[
a = 0 \land (a := a-1. \ x := (a+b)! / (a!b!). \ a := a+1. \ x := x \times (a+b)/a) \quad \text{assignment}
\]

\[
= a = 0 \land (a := a-1. \ x := (a+b)! / (a!b!). \ a := a+1. \ x := x \times (a+b)/a \land a' = a \land b' = b)
\]

\[
= a = 0 \land x' = (a-1+b)! / ((a-1)!b!) \times (a+b)/a \land a' = a \land b' = b \quad \text{substitution law 3 times}
\]

\[
= a = 0 \land x' = (a+b)! / (a!b!) \land a' = a \land b' = b \quad \text{simplify}
\]

\[
= a = 0 \land x' := (a+b)! / (a!b!) \land a' = a \land b' = b \quad \text{specialization}
\]

\[
\Rightarrow x := (a+b)! / (a!b!)
\]

Now the time.

\[
t' = t+a \iff \text{if } a = 0 \text{ then } x := 1
\]
\[
\text{else } a := a-1. \ t := t+1. \ t' = t+a. \ a := a+1. \ x := x \times (a+b)/a \ \
\]

Proof by cases. First case:

\[
a = 0 \land (x := 1) \Rightarrow t' = t+a \quad \text{definition of assignment}
\]

\[
= a = 0 \land x' = 1 \land a' = a \land b' = b \land t' = t \Rightarrow t' = t+a
\]

\[
= T
\]

Second case, starting with the right side:

\[
a = 0 \land (a := a-1. \ t := t+1. \ t' = t+a. \ a := a+1. \ x := x \times (a+b)/a) \quad \text{assignment}
\]

\[
= a = 0 \land (a := a-1. \ t := t+1. \ t' = t+a. \ a := a+1. \ x := x \times (a+b)/a \land a' = a \land b' = b \land t' = t)
\]

\[
= a = 0 \land (t' = t+a. \ x' := x \times (a+1+b)/(a+1) \land a' = a+1 \land b' = b \land t' = t) \quad \text{substitution law 3 times}
\]

\[
= a = 0 \land (\exists x', a', b', t'. \ t' = t+a \land x' := x \times (a+1+b)/(a+1) \land a' = a+1 \land b' = b \land t' = t') \quad \text{sequential composition}
\]

\[
= a = 0 \land (\forall x'', a'', b'', t''. \ t'' = t+a \land x'' := x \times (a+1+b)/(a+1) \land a' = a+1 \land b' = b \land t = t'') \quad \text{one point 4 times}
\]

\[
= a = 0 \land t' = t+a \quad \text{specialization}
\]

\[
\Rightarrow t' = t+a
\]

When refining \(x := (a+b)! / (a!b!)\), there was no time variable. Adding the time variable, we cannot write this as an assignment, because that would mean \(t' = t\). We can put the result and the timing together as

\[
x' := (a+b)! / (a!b!) \land a' = a \land b' = b \land t' = t+a
\]

or as

\[
x := (a+b)! / (a!b!). \ t := t+a
\]

Here is a solution that is symmetric in \(a\) and \(b\).

\[
x := (a+b)! / (a!b!) \iff
\]

\[
\text{if } a = 0 \lor b = 0 \text{ then } x := 1
\]
\[
\text{else } a := a-1. \ b := b-1. \ x := (a+b)! / (a!b!). \ \
a := a+1. \ b := b+1. \ x := x / a / b \times (a+b-1) \times (a+b) \ \
\]

And its execution time is smaller: \(a \downarrow b\).

Here is a solution with the same execution time and its recursion does not require a stack.

\[
x' = (a+b)! / (a!b!) \land t' = t + a \downarrow b \iff
\]
\[
x := 1. \ x' = x \times (a+b)! / (a!b!) \land t' = t + a \downarrow b
\]
\[ x' = x \times (a+b)! / (a\times b!) \land t' = t + a \downarrow b \iff \]
\[ \begin{array}{l}
\text{if } a=0 \lor b=0 \text{ then } \text{ok} \\
\text{else } \text{ if } a:= x/a/b \times (a+b-1) \times (a+b). \ a:= a-1. \ b:= b-1. \ t:= t+1. \\
\text{then } x' = x \times (a+b)! / (a\times b!) \land t' = t + a \downarrow b \text{ fi}
\end{array} \]

Now, here is a for-loop solution. Define invariant
\[ A k \iff x = (a+k)! / (a\times k)! \]
Then
\[ \begin{array}{l}
x' = (a+b)! / (a\times b!) \iff x:= 1. \ A 0 \Rightarrow A'b \\
A 0 \Rightarrow A'b \iff \text{ for } k:= 0;..b \text{ do } k:= 0;..b \land A k \Rightarrow A'(k+1) \text{ od} \\
k: 0;..b \land A k \Rightarrow A'(k+1) \iff x:= x\times(a+k+1)/(k+1)
\end{array} \]
with timing \[ t' = t + b \cdot \]

Finally, here are two functional solutions. Define
\[ f = \langle a, b: \text{nat} \cdot (a+b)! / (a\times b)! \rangle \]
Then
\[ \begin{array}{l}
f a b = \begin{cases} 1 & \text{if } a=0 \\ f(a-1) \times (a+b) / a & \text{else} \end{cases} \\
\text{with execution time } a. \ \text{For execution time } a \downarrow b \\
f a b = \begin{cases} 1 & \text{if } a=0 \lor b=0 \\ f(a-1) \times (b-1) \times (a+b-1) \times (a+b) / a \times b & \text{else} \end{cases}
\end{array} \]