The number of ways to partition \( a+b \) things into \( a \) things and \( b \) things is \((a+b)! / (a!b!)\) where \( ! \) is the factorial function. First without time.

\[
x := (a+b)! / (a!b!) \iff
\]

- if \( a=0 \) then \( x := 1 \)
- else \( a := a-1 \).

\( x := (a+b)! / (a!b!) \). \( a := a+1 \).

\( x := x \times (a+b) / a \) \textbf{fi}

The assignment \( x := (a+b)! / (a!b!) \) means \( x' = (a+b)! / (a!b!) \) \( \land a' = a \land b' = b \). On the right side it is a recursive call. Stating it as an assignment makes the proof easy: just use the substitution law and simplify. The proof is by cases. First case:

\[
a = 0 \land (x := 1) \Rightarrow (x := (a+b)! / (a!b!)) \quad \text{definition of assignment}
\]

\[
\equiv a = 0 \land x' = 1 \land a' = a \land b' = b \Rightarrow x' = (a+b)! / (a!b!) \land a' = a \land b' = b \quad \text{use 0!} = 1
\]

\[
\equiv \top
\]

Second case, starting with the right side:

\[
a = 0 \land x' = (a-1+b)! / ((a-1)!b!) \times (a+b)/a \land a' = a \land b' = b
\]

\[
\equiv a = 0 \land x' = (a+b)! / (a!b!) \land a' = a \land b' = b \quad \text{simplify}
\]

\[
\equiv a = 0 \land x' = (a+b)! / (a!b!) \land a' = a \land b' = b \quad \text{specialization}
\]

\[
\Rightarrow x := (a+b)! / (a!b!)
\]

Now the time.

\[
t' = t+a \iff
\]

- if \( a=0 \) then \( x := 1 \)
- else \( a := a-1 \).

\( t := t+1 \). \( t' = t+a \). \( a := a+1 \). \( x := x \times (a+b) / a \) \textbf{fi}

Proof by cases. First case:

\[
a = 0 \land (x := 1) \Rightarrow t' = t+a \quad \text{definition of assignment}
\]

\[
\equiv a = 0 \land x' = 1 \land a' = a \land b' = b \Rightarrow t' = t+a
\]

\[
\equiv \top
\]

Second case, starting with the right side:

\[
a = 0 \land (t' = t+a) \land x' = x \times (a+1+b) / (a+1) \land a' = a+1 \land b' = b \land t' = t
\]

\[
\equiv a = 0 \land (t' = t+a) \land x' = x \times (a+1+b) / (a+1) \land a' = a+1 \land b' = b \land t' = t
\]

\[
\equiv a = 0 \land (\exists x'', a'', b'', t'') \cdot t'' = t+a \land x'' = x' \times (a''+1+b'') / (a''+1)
\]

\[
\land a'' = a'+1 \land b'' = b' \land t'' = t'
\]

\[
\equiv a = 0 \land t' = t+a \quad \text{one point 4 times}
\]

\[
\Rightarrow t' = t+a
\]

When refining \( x := (a+b)! / (a!b!) \), there was no time variable. Adding the time variable, we cannot write this as an assignment, because that would mean \( t' = t \). We can put the result and the timing together as

\[
x' = (a+b)! / (a!b!) \land a' = a \land b' = b \land t' = t+a
\]

or as

\[
x := (a+b)! / (a!b!). \ t := t+a
\]

Here is a solution that is symmetric in \( a \) and \( b \).

\[
x := (a+b)! / (a!b!) \iff
\]

- if \( a = 0 \lor b = 0 \) then \( x := 1 \)
- else \( a := a-1 \). \( b := b-1 \).

\( x := (a+b)! / (a!b!) \).

\[
\equiv a := a+1 \land b := b+1 \land x := x/a \times (a+b-1) \times (a+b)
\]

And its execution time is smaller: \( a \downarrow b \).
Here is a solution with the same execution time and its recursion does not require a stack.

\[
x' = \frac{(a+b)!}{(a!b)!} \land t' = t + a \downarrow b \iff \\
x := 1. \; x' = x \times (a+b)! / (a!b)! \land t' = t + a \downarrow b \\
x' = x \times (a+b)! / (a!b)! \land t' = t + a \downarrow b \iff \\
\begin{cases} 
\text{if } a=0 \lor b=0 \text{ then ok} \\
\text{else } x := x / a / b \times (a+b-1) \times (a+b). \; a := a-1. \; b := b-1. \; t := t+1. \\
x' = x \times (a+b)! / (a!b)! \land t' = t + a \downarrow b \end{cases}
\]

Now, here is a for-loop solution. Define invariant

\[
A_k = x = \frac{(a+k)!}{(a)!k!}
\]

Then

\[
x' = (a+b)! / (a!b)! \iff x := 1. \; A_0 \Rightarrow A' b \\
A_0 \Rightarrow A' b \iff \text{for } k := 0..b \text{ do } k := 0..b \land A_k \Rightarrow A'(k+1) \text{ od} \\
k := 0..b \land A_k \Rightarrow A'(k+1) \iff x := x \times (a+k+1)/(k+1)
\]

with timing \( t' = t+b \).

Finally, here are two functional solutions. Define

\[
f = \langle a, b: \text{nat} \rightarrow (a+b)! / (a!b)! \rangle
\]

Then

\[
f a b = \begin{cases} 
1 & \text{if } a=0 \\
\text{else } f(a-1) \times (a+b) / a \end{cases}
\]

with execution time \( a \). For execution time \( a \downarrow b \)

\[
f a b = \begin{cases} 
1 & \text{if } a=0 \lor b=0 \\
\text{else } f(a-1) \times (b-1) \times (a+b-1) \times (a+b) / a / b \end{cases}
\]