173 (list comparison) Using item comparison but not list comparison, write a program to determine whether one list comes before another in the list order.

After trying the question, scroll down to the solution.

§ Let the lists be L and M. We need binary variable s to record the answer, and natural variable n as index.

$$s' = (L<M) \quad \Longleftarrow \quad n:= 0.\ \ n\le\#L \wedge n\le\#M \ \Rightarrow\ s' = (L[n;..\#L] < M[n;..\#M])$$

$$\begin{array}{l} n\le\#L \wedge n\le\#M \ \Rightarrow\ s' = (L[n;..\#L] < M[n;..\#M]) \quad \Longleftarrow \\ \qquad \mathbf{if}\ n{=}\#M\ \mathbf{then}\ s:= \bot \\ \qquad \mathbf{else\ if}\ n{=}\#L\ \mathbf{then}\ s:= \top \\ \qquad\quad \mathbf{else\ if}\ L\,n < M\,n\ \mathbf{then}\ s:= \top \\ \qquad\qquad \mathbf{else\ if}\ L\,n > M\,n\ \mathbf{then}\ s:= \bot \\ \qquad\qquad\quad \mathbf{else}\ n:= n{+}1. \\ \qquad\qquad\qquad n\le\#L \wedge n\le\#M \ \Rightarrow\ s' = (L[n;..\#L] < M[n;..\#M])\ \mathbf{fi\ fi\ fi\ fi} \end{array}$$

First refinement, right side:

$$\begin{array}{lll} & n:= 0.\ \ n\le\#L \wedge n\le\#M \ \Rightarrow\ s' = (L[n;..\#L] < M[n;..\#M]) & \text{substitution law} \\ = & 0\le\#L \wedge 0\le\#M \ \Rightarrow\ s' = (L[0;..\#L] < M[0;..\#M]) & \\ = & s' = (L<M) & \end{array}$$

Last refinement, by cases. First case:

$$\begin{array}{lll} & (n\le\#L \wedge n\le\#M \ \Rightarrow\ s' = (L[n;..\#L] < M[n;..\#M])) \ \Leftarrow\ n{=}\#M \wedge (s:= \bot) & \\ & & \text{portation, expand assignment} \\ = & n{=}\#M{\le}\#L \wedge s'{=}\bot \wedge n'{=}n \ \Rightarrow\ s' = (L[n;..\#L] < M[n;..\#M]) & \text{context} \\ = & n{=}\#M{\le}\#L \wedge s'{=}\bot \wedge n'{=}n \ \Rightarrow\ L[\#M;..\#L] \ge M[\#M;..\#M] & (\text{any list}){\ge}[nil] \\ = & \top & \end{array}$$

Second case:

$$\begin{array}{lll} & (n\le\#L \wedge n\le\#M \ \Rightarrow\ s' = (L[n;..\#L] < M[n;..\#M])) \ \Leftarrow\ n{\ne}\#M \wedge n{=}\#L \wedge (s:= \top) & \\ & & \text{portation, expand assignment} \\ = & n{=}\#L{<}\#M \wedge s'{=}\top \wedge n'{=}n \ \Rightarrow\ s' = (L[n;..\#L] < M[n;..\#M]) & \text{context} \\ = & n{=}\#L{<}\#M \wedge s'{=}\top \wedge n'{=}n \ \Rightarrow\ L[\#L;..\#L] < M[\#L;..\#M] & [nil]{<}(\text{any nonempty list}) \\ = & \top & \end{array}$$

Middle case:

$$\begin{array}{lll} & \quad (n\le\#L \wedge n\le\#M \ \Rightarrow\ s' = (L[n;..\#L] < M[n;..\#M])) & \\ & \Leftarrow\ \ n{\ne}\#M \wedge n{\ne}\#L \wedge L\,n < M\,n \wedge (s:= \top) & \text{portation, expand assignment} \\ = & n{<}\#L \wedge n{<}\#M \wedge L\,n < M\,n \wedge s' \wedge n'{=}n \ \Rightarrow\ s' = (L[n;..\#L] < M[n;..\#M]) & \text{context } s' \\ = & n{<}\#L \wedge n{<}\#M \wedge L\,n < M\,n \wedge s' \wedge n'{=}n \ \Rightarrow\ L[n;..\#L] < M[n;..\#M] & \\ & & \text{weaken antecedent} \\ \Leftarrow & n{<}\#L \wedge n{<}\#M \wedge L\,n < M\,n \ \Rightarrow\ L[n;..\#L] < M[n;..\#M] & \text{the segments are nonempty} \\ & \text{and the first item of } L[n;..\#L] \text{ is less than the first item of } M[n;..\#M] & \\ = & \top & \end{array}$$

Next-to-last case:

$$\begin{array}{lll} & \quad (n\le\#L \wedge n\le\#M \ \Rightarrow\ s' = (L[n;..\#L] < M[n;..\#M])) & \\ & \Leftarrow\ \ n{\ne}\#M \wedge n{\ne}\#L \wedge L\,n > M\,n \wedge (s:= \bot) & \text{portation, expand assignment} \\ = & n{<}\#L \wedge n{<}\#M \wedge L\,n > M\,n \wedge s'{=}\bot \wedge n'{=}n \ \Rightarrow\ s' = (L[n;..\#L] < M[n;..\#M]) & \\ & & \text{context } s' \\ = & n{<}\#L \wedge n{<}\#M \wedge L\,n > M\,n \wedge \neg s' \wedge n'{=}n \ \Rightarrow\ L[n;..\#L] \ge M[n;..\#M] & \\ & & \text{weaken antecedent} \\ \Leftarrow & n{<}\#L \wedge n{<}\#M \wedge L\,n > M\,n \ \Rightarrow\ L[n;..\#L] \ge M[n;..\#M] & \text{the segments are nonempty} \\ & \text{and the first item of } L[n;..\#L] \text{ is greater than the first item of } M[n;..\#M] & \\ = & \top & \end{array}$$

Last case:

$$
\begin{aligned}
& (n{\le}\#L \wedge n{\le}\#M \;\Rightarrow\; s' = (L[n;..\#L] < M[n;..\#M])) \\
\Leftarrow\quad & n{\ne}\#M \wedge n{\ne}\#L \wedge L\,n = M\,n \\
& \wedge (n{:=}\,n{+}1.\;\; n{\le}\#L \wedge n{\le}\#M \;\Rightarrow\; s' = (L[n;..\#L] < M[n;..\#M])) && \text{portation, substitution} \\
=\quad & n{<}\#L \wedge n{<}\#M \wedge L\,n = M\,n \\
& \wedge (n{+}1{\le}\#L \wedge n{+}1{\le}\#M \;\Rightarrow\; s' = (L[n{+}1;..\#L] < M[n{+}1;..\#M])) \\
& \Rightarrow\; s' = (L[n;..\#L] < M[n;..\#M]) && \text{discharge} \\
=\quad & n{<}\#L \;\wedge\; n{<}\#M \;\wedge\; Ln{=}Mn \;\wedge\; s' = (L[n{+}1;..\#L] < M[n{+}1;..\#M]) \\
& \Rightarrow\; s' = (L[n;..\#L] < M[n;..\#M]) && \text{context } s' \text{ and weaken antecedent} \\
=\quad & n{<}\#L \;\wedge\; n{<}\#M \;\wedge\; L\,n = M\,n \\
& \Rightarrow\; (L[n{+}1;..\#L] < M[n{+}1;..\#M]) = (L[n;..\#L] < M[n;..\#M]) && \text{the segments are nonempty and their first items are equal so the remaining items determine the order of the segments} \\
=\quad & \top
\end{aligned}
$$