157 (Zeno) Here is a loop.

$$R \Leftarrow x:= x+1.\ R$$

Suppose we charge time 2^{-x} for the recursive call, so that each iteration takes half as long as the one before. Prove that the execution time is finite.

After trying the question, scroll down to the solution.

§ I prove

$$t' = t+2^{-x} \Leftarrow x := x+1.\ t := t+2^{-x}.\ t' = t+2^{-x}$$

starting with the right side:

	$x := x+1.\ t := t+2^{-x}.\ t' = t+2^{-x}$	Substitution Law
$=$	$x := x+1.\ t' = t+2^{-x}+2^{-x}$	arithmetic
$=$	$x := x+1.\ t' = t+2^{-(x-1)}$	Substitution Law
$=$	$t' = t+2^{-(x+1-1)}$	arithmetic
$=$	$t' = t+2^{-x}$	

The refinement with Zeno time

$$R \Leftarrow x := x+1.\ t := t+2^{-x}.\ R$$

can be proven for many different specifications R . Here are four.

$t' = t+2^{-x}$

$t'=\infty$

$x'=3$

$x'=4$

It seems very odd that the execution time can be either 2^{-x} or ∞ . Can't we start with $x=0$ and $t=0$ and observe whether the computation ends at time 1 or takes longer? And if it does end at time 1 , can't we see if the final value of x is 3 or 4 or something else? Since we cannot build a physical mechanism such that each iteration takes half as long as the previous iteration, we cannot observe what would happen. But mathematically, can't we determine which answer is right? If x and t start at 0 , then $t' = \Sigma n\colon nat+1\cdot\ 2^{-n}$.

	$2\times t'$	
$=$	$2 \times \Sigma n\colon nat+1\cdot\ 2^{-n}$	distribute multiplication over sum
$=$	$\Sigma n\colon nat+1\cdot\ 2^{-n+1}$	quantifier law
$=$	$2^{-1+1} + \Sigma n\colon nat+2\cdot\ 2^{-n+1}$	simplify and change of variable
$=$	$1 + \Sigma n\colon nat+1\cdot\ 2^{-n}$	
$=$	$1+t'$	

Solving $2\times t' = 1+t'$, we find 3 solutions: $t'=-\infty$, $t'=1$, and $t'=\infty$. Since time cannot be negative, that leaves two possibilities: either the final time is 1 or it is ∞ .

After a finite number m of iterations, the execution time is $1-2^{-m}$, which has limit 1 .

	$\Updownarrow m\cdot\ \Sigma n\colon (0,..m)+1\cdot\ 2^{-n}$
$=$	$\Updownarrow m\cdot\ 1-2^{-m}$
$=$	1

The fact that this limit is 1 does not say $t'=1$. For that, we need (but don't have) one further property, called continuity, that allows us to move $\Updownarrow$ inwards.

	$\Updownarrow m\cdot\ \Sigma n\colon (0,..m)+1\cdot\ 2^{-n}$	now use continuity
$=$	$\Sigma n\colon (0,..(\Updownarrow m\cdot\ m))+1\cdot\ 2^{-n}$	
$=$	$\Sigma n\colon (0,..\infty)+1\cdot\ 2^{-n}$	
$=$	t'	

If we had continuity, we could conclude $t'=1$ when $x=t=0$, or more generally, $t' = t+2^{-x}$.

I said earlier "we cannot build a physical mechanism such that each iteration takes half as long as the previous iteration". But, according to current physics, nature has built such a mechanism; it is called a black hole. To an outside observer watching an object fall into a black hole, the object takes forever to reach the event horizon (a specific distance from the center of the black hole), and it never gets past that horizon. But to someone who is falling into a black hole, they reach the event horizon in finite time and continue falling inward. So which is it: finite or infinite time to reach the event horizon? The answer depends upon your observation point. Are you watching from outside, or are you the object falling in?

The previous paragraph is standard current physics, but I have doubts. See Time Dilation.