152 Let n be natural and let i and j be natural variables. Here are two refinements.

$$A \Leftarrow i:= 0.\ j:= n.\ B$$

$$B \Leftarrow \textbf{if } i\ge j \textbf{ then } ok \textbf{ else } i:= i+1.\ j:= j-1.\ B \textbf{ fi}$$

(a) Add recursive time.

(b) Find specifications A and B that give good upper bounds on the time, and prove the refinements.

After trying the question, scroll down to the solution.

(a) Add recursive time.

§ $A \Leftarrow i{:=}\,0.\ \ j{:=}\,n.\ \ B$

$B \Leftarrow \textbf{if } i{\ge}j \textbf{ then } ok \textbf{ else } i{:=}\,i{+}1.\ \ j{:=}\,j{-}1.\ \ t{:=}\,t{+}1.\ \ B \textbf{ fi}$

(b) Find specifications A and B that give good upper bounds on the time, and prove the refinements.

§ I first tried

$A = t' \le t + (n{+}1)/2$

$B = t' \le t + \textbf{if } i{\ge}j \textbf{ then } 0 \textbf{ else } (j{-}i{+}1)/2 \textbf{ fi}$

but it didn't work, so now I'll try

$A = \ \ (odd\ n \Rightarrow t' = t + (n{+}1)/2)$
$\qquad \wedge\ (even\ n \Rightarrow t' = t + n/2)$

$B = \ \ (i{\le}j{+}1 \wedge odd\ (j{-}i) \Rightarrow t' = t + (j{-}i{+}1)/2)$
$\qquad \wedge\ (i{\le}j \wedge even\ (j{-}i) \Rightarrow t' = t + (j{-}i)/2)$

Proof of first refinement:

$i{:=}\,0.\ \ j{:=}\,n.\ \ B$ — replace B and substitute twice

$= \ \ (0{\le}n \wedge odd\ (n{-}0) \Rightarrow t' = t + (n{-}0{+}1)/2)$
$\quad \wedge\ (0{\le}n \wedge even\ (n{-}0) \Rightarrow t' = t + (n{-}0)/2)$ — simplify

$= \ \ (odd\ n \Rightarrow t' = t + (n{+}1)/2)$
$\quad \wedge\ (even\ n \Rightarrow t' = t + n/2)$

$= A$

Proof of last refinement, starting with the right side:

$\textbf{if } i{\ge}j \textbf{ then } ok \textbf{ else } i{:=}\,i{+}1.\ \ j{:=}\,j{-}1.\ \ t{:=}\,t{+}1.\ \ B \textbf{ fi}$ — replace B and substitute thrice

$= \textbf{if } i{\ge}j \textbf{ then } ok \textbf{ else } \ (i{+}1{\le}j{-}1{+}1 \wedge odd\ (j{-}1{-}(i{+}1)) \Rightarrow t' = t + 1 + (j{-}1{-}(i{+}1){+}1)/2)$
$\qquad \wedge\ (i{+}1{\le}j{-}1 \wedge even\ (j{-}1{-}(i{+}1)) \Rightarrow t' = t + 1 + (j{-}1{-}(i{+}1))/2) \textbf{ fi}$

$= \textbf{if } i{\ge}j \textbf{ then } ok \textbf{ else } \ (i{\le}j{-}1 \wedge odd\ (j{-}i) \Rightarrow t' = t + (j{-}i{+}1)/2)$
$\qquad \wedge\ (i{\le}j{-}2 \wedge even\ (j{-}i) \Rightarrow t' = t + (j{-}i)/2) \textbf{ fi}$

In context $i{<}j$ and $odd\ (j{-}i)$, $i{\le}j{-}1$ is the same as $i{\le}j$.
In context $i{<}j$ and $even\ (j{-}i)$, $i{\le}j{-}2$ is the same as $i{\le}j$.

$= \textbf{if } i{\ge}j \textbf{ then } ok \textbf{ else } \ (i{\le}j \wedge odd\ (j{-}i) \Rightarrow t' = t + (j{-}i{+}1)/2)$
$\qquad \wedge\ (i{\le}j \wedge even\ (j{-}i) \Rightarrow t' = t + (j{-}i)/2) \textbf{ fi}$

$= \textbf{if } i{\ge}j \textbf{ then } ok \textbf{ else } B \textbf{ fi}$

In context $i{\ge}j$ and $odd\ (j{-}i)$, $i{\le}j{+}1$ is the same as $i{=}j{+}1$.
In context $i{\ge}j$ and $even\ (j{-}i)$, $i{\le}j$ is the same as $i{=}j$.

$= \textbf{if } i{\ge}j \textbf{ then } \ (i{\le}j{+}1 \wedge odd\ (j{-}i) \Rightarrow i'{=}i \wedge j'{=}j \wedge t' = t + (j{-}i{+}1)/2)$
$\qquad \wedge\ (i{\le}j \wedge even\ (j{-}i) \Rightarrow i'{=}i \wedge j'{=}j \wedge t' = t + (j{-}i)/2)$
$\quad \textbf{else } B \textbf{ fi}$

$\Rightarrow \textbf{if } i{\ge}j \textbf{ then } B \textbf{ else } B \textbf{ fi}$

$= B$