Let $x$ be an integer variable. Let $P$ be a specification refined as follows.

$$
P \iff \begin{cases} 
  x > 0 & \text{then } x := x - 2. \ P \\
  x < 0 & \text{then } x := x + 1. \ P \\
  \text{else ok} & \text{fi}
\end{cases}
$$

(a) Prove the refinement when $P = x' = 0$.

§ Using Refinement by Cases, I must prove three things:

1. $x' = 0 \iff x > 0 \land (x := x - 2. \ x' = 0)$
2. $x' = 0 \iff x < 0 \land (x := x + 1. \ x' = 0)$
3. $x' = 0 \iff x = 0 \land \text{ok}$

Let's start with the first.

$x > 0 \land (x := x - 2. \ x' = 0) \Rightarrow x' = 0$

use substitution law

Now the middle one.

$x < 0 \land (x := x + 1. \ x' = 0) \Rightarrow x' = 0$

use substitution law

And the last one.

$x = 0 \land \text{ok} \Rightarrow x' = x$

transitivity

(b) Add recursive time and find and prove an upper bound for the execution time.

§ Adding recursive time,

$$
P \iff \begin{cases} 
  x > 0 & \text{then } x := x - 2. \ t := t + 1. \ P \\
  x < 0 & \text{then } x := x + 1. \ t := t + 1. \ P \\
  \text{else ok} & \text{fi}
\end{cases}
$$

The exact execution timing specification is

$$
P = t' = t + \text{if } x > 0 \text{ then } \text{ceil}(x/2) + 2 - \text{mod} x 2 \text{ else } -x \text{ fi}
$$

but ceil and mod are awkward functions to deal with, so I'll prove

$$
P = \text{if } x > 0 \text{ then } t' \leq t + x/2 + 2 \text{ else } t' \leq t-x \text{ fi}
$$

(I tried re-expressing $P$ as a conjunction

$$
P = (x > 0 \Rightarrow t' \leq t + x/2 + 2) \land (x \leq 0 \Rightarrow t' \leq t-x)
$$

so that I can use Refinement by Parts, but that didn't work. That's because the $x > 0$ part may take $x$ to 0 or below, and require the other part.)

Using Refinement by Cases, I must prove three things:

1. $x > 0 \land (x := x - 2. \ t := t + 1. \ P) \Rightarrow P$
2. $x < 0 \land (x := x + 1. \ t := t + 1. \ P) \Rightarrow P$
3. $x = 0 \land \text{ok} \Rightarrow P$

Let's start with the first case.

$x > 0 \land (x := x - 2. \ t := t + 1. \ P) \Rightarrow P$

replace first $P$

Substitution Law twice

$x > 0 \land \text{if } x = 2 \text{ then } t' \leq t + 1 + (x-2)/2 + 2 \text{ else } t' \leq t + 1 - (x-2) \text{ fi} \Rightarrow P$

simplify

note that $x > 0 \equiv x = 1 \lor x = 2 \lor x > 2$ and then distribute
\[
\begin{align*}
\text{Now the last case.} & \\
(x=0 \land \text{ok} \Rightarrow P) & \quad \text{replace } \text{ok and } P \\
\land (x=0 \land x' = x \land t' \leq t \Rightarrow \text{if } x>0 \text{ then } t' \leq t + x/2 + 2 \text{ else } t' \leq t-x \text{ fi}) & \quad \text{context} \\
\land (x=0 \land x' = x \land t' \leq t \Rightarrow t \leq t-0) & \quad \text{arithmetic, reflexive, base} \\
\end{align*}
\]