135 Let x and n be natural variables. Find a specification P such that both the following refinements can be proven:

$$x = x' \times 2^{n'} \Leftarrow n := 0.\ P$$

$$P \Leftarrow \textbf{if } even\ x \textbf{ then } x := x/2.\ n := n+1.\ P \textbf{ else } ok \textbf{ fi}$$

After trying the question, scroll down to the solution.

§ $P \;=\; x = x'\times 2^{n'-n}$

Proof:

$$n{:=}\,0.\;\; x = x'\times 2^{n'-n} \qquad \text{Substitution Law}$$
$$=\;\; x = x'\times 2^{n'}$$

The second one is proven by cases. First,

$$\mathit{even}\ x \;\wedge\; (x{:=}\,x/2.\;\; n{:=}\,n{+}1.\;\; x = x'\times 2^{n'-n}) \qquad \text{use Substitution Law twice}$$
$$=\;\; \mathit{even}\ x \;\wedge\; x/2 = x'\times 2^{n'-(n+1)} \qquad \text{number theory}$$
$$=\;\; \mathit{even}\ x \;\wedge\; x = x'\times 2^{n'-n} \qquad \text{specialization}$$
$$\Longrightarrow\;\; P$$

Now the other case:

$$(x = x'\times 2^{n'-n} \;\Longleftarrow\; \neg\, \mathit{even}\ x \;\wedge\; \mathit{ok}) \qquad \text{expand } \mathit{ok}$$
$$=\;\; (x = x'\times 2^{n'-n} \;\Longleftarrow\; \neg\, \mathit{even}\ x \;\wedge\; x'{=}x \;\wedge\; n'{=}n)$$

use antecedent as context in consequent

$$=\;\; (x = x\times 2^{n-n} \;\Longleftarrow\; \neg\, \mathit{even}\ x \;\wedge\; x'{=}x \;\wedge\; n'{=}n) \qquad \text{number theory}$$
$$=\;\; (\top \;\Longleftarrow\; \neg\, \mathit{even}\ x \;\wedge\; x'{=}x \;\wedge\; n'{=}n) \qquad \text{base law for } \Longleftarrow$$
$$=\;\; \top$$