Let $x$ and $n$ be natural variables. Find a specification $P$ such that both the following refinements can be proven:

$$x = x' \times 2^n \iff n := 0. \ P$$

$$P \iff \text{if even } x \text{ then } x := x/2. \ n := n + 1. \ P \text{ else ok fi}$$

§

$$P \equiv x = x' \times 2^{n'-n}$$

Proof:

$$\begin{align*}
\quad & n := 0. \ x = x' \times 2^{n'-n} \\
\equiv & x = x' \times 2^{n'}
\end{align*}$$

Substitution Law

The second one is proven by cases. First,

$$\begin{align*}
\quad & \text{even } x \land (x := x/2. \ n := n + 1. \ x = x' \times 2^{n'-n}) \\
\equiv & \text{even } x \land x/2 = x' \times 2^{n'-n + 1)} \\
\equiv & \text{even } x \land x = x' \times 2^{n'-n}
\end{align*}$$

use Substitution Law twice

number theory

specialization

$\Rightarrow P$

Now the other case:

$$\begin{align*}
\quad & (x = x' \times 2^{n'-n} \iff \neg \text{even } x \land \text{ok}) \\
\equiv & (x = x' \times 2^{n'-n} \iff \neg \text{even } x \land x' = x \land n' = n) \\
\equiv & (x = x \times 2^{n-n} \iff \neg \text{even } x \land x' = x \land n' = n) \\
\equiv & (\top \iff \neg \text{even } x \land x' = x \land n' = n) \\
\equiv & \top
\end{align*}$$

base law for $\iff$