For what exact precondition and postcondition does the following assignment move integer variable \( x \) farther from zero staying on the same side of zero?

§ What does “staying on the same side of zero” mean if the initial value of \( x \) is zero? Since that’s not clear, let’s say that \( x’ \) can be on either side in that case. The specification is

\[
(x<0 \Rightarrow x’<0) \land (x=0 \Rightarrow x’=0) \land (x>0 \Rightarrow x’>0)
\]

\[
x’<x <0 \lor x’+x = 0 \lor x’>x >0
\]

(a) \( x := x + 1 \)

§ (exact precondition for \( x’<x <0 \lor x’+x = 0 \lor x’>x >0 \) to be refined by \( x := x + 1 \))

\[
\forall x’: x’<x <0 \lor x’+x = 0 \lor x’>x >0 \Rightarrow (x := x + 1)
\]

\[
\forall x’: x’<x <0 \lor x’+x = 0 \lor x’>x >0 \Rightarrow x’=x + 1 \quad \text{One-point}
\]

\[
x +1<x <0 \lor x +1+x = 0 \lor x +1>x >0
\]

\[
\bot \lor x = 0 \lor x >0
\]

\[
x \geq 0
\]

We can be sure that \( x := x + 1 \) will move \( x \) farther from zero, staying on the same side, if \( x \geq 0 \).

(b) \( x := \text{abs}(x + 1) \)

§ (exact precondition for \( x’<x <0 \lor x’+x = 0 \lor x’>x >0 \) to be refined by \( x := \text{abs}(x + 1) \))

\[
\forall x’: x’<x <0 \lor x’+x = 0 \lor x’>x >0 \Rightarrow (x := \text{abs}(x + 1))
\]

\[
\forall x’: x’<x <0 \lor x’+x = 0 \lor x’>x >0 \Rightarrow x’=\text{abs}(x + 1) \quad \text{One-point}
\]

\[
\text{abs}(x + 1)<x <0 \lor \text{abs}(x + 1)+x = 0 \lor \text{abs}(x + 1)>x >0
\]

\[
\bot \lor x = 0 \lor x >0
\]

\[
x \geq 0
\]

We can be sure that \( x := \text{abs}(x + 1) \) will move \( x \) farther from zero, staying on the same side, if \( x \geq 0 \).

(c) \( x := x^2 \)

§ (exact precondition for \( x’<x <0 \lor x’+x = 0 \lor x’>x >0 \) to be refined by \( x := x^2 \))

\[
\forall x’: x’<x <0 \lor x’+x = 0 \lor x’>x >0 \Rightarrow (x := x^2)
\]

\[
\forall x’: x’<x <0 \lor x’+x = 0 \lor x’>x >0 \Rightarrow x’=x^2 \quad \text{One-point}
\]
We can be sure that \( x := x^2 \) will move \( x \) farther from zero, staying on the same side, if \( x \geq 2 \).

(exact postcondition for \( x' < 0 \lor x' > 0 \) to be refined by \( x := x^2 \))

\[
\begin{align*}
\forall x' \cdot x' < 0 & \lor x' = 0 \lor x' > 0 \iff (x := x^2) \\
\forall x' \cdot x^2 < 0 & \lor x^2 + 1 = 0 \lor x^2 > 0 \iff x' = x^2 \\
\forall x' \cdot x \leq 1 & \iff x' = x^2 \\
\forall x' \cdot x \leq 1 & \iff x' = x^2 \\
\forall x' \cdot x \leq 1 & \iff x' = x^2 \\
\forall x' \cdot x \leq 1 & \iff x' = x^2
\end{align*}
\]

We can be sure that \( x := x^2 \) moved \( x \) farther from zero, staying on the same side, if \( x' \) is not a square. But of course it will be a square, so we can never be sure that \( x \) moved farther from zero, staying on the same side.