127 Prove or disprove

(a) $R.\ \textbf{if } b \textbf{ then } P \textbf{ else } Q \textbf{ fi} \;=\; \textbf{if } b \textbf{ then } R.\,P \textbf{ else } R.\,Q \textbf{ fi}$

(b) $\textbf{if } b \textbf{ then } P{\Rightarrow}Q \textbf{ else } R{\Rightarrow}S \textbf{ fi} \;=\; \textbf{if } b \textbf{ then } P \textbf{ else } R \textbf{ fi} \Rightarrow \textbf{if } b \textbf{ then } Q \textbf{ else } S \textbf{ fi}$

(c) $\textbf{if } b \textbf{ then } P.\,Q \textbf{ else } R.\,S \textbf{ fi} \;=\; \textbf{if } b \textbf{ then } P \textbf{ else } R \textbf{ fi}.\ \textbf{if } b \textbf{ then } Q \textbf{ else } S \textbf{ fi}$

After trying the question, scroll down to the solution.

(a) $R.\ \textbf{if}\ b\ \textbf{then}\ P\ \textbf{else}\ Q\ \textbf{fi}\ \ =\ \ \textbf{if}\ b\ \textbf{then}\ R.\,P\ \textbf{else}\ R.\,Q\ \textbf{fi}$

§ Let binary variable b be the only variable, let it initially be $\bot$, and let $R = b{:=}\top$, let $P = b{:=}\top$, let $Q = b{:=}\bot$. Then

$$\begin{array}{ll} & R.\ \textbf{if}\ b\ \textbf{then}\ P\ \textbf{else}\ Q\ \textbf{fi} \\ = & b{:=}\top.\ \textbf{if}\ b\ \textbf{then}\ b{:=}\top\ \textbf{else}\ b{:=}\bot\ \textbf{fi} \\ = & \textbf{if}\ b{:=}\top.\ b\ \textbf{then}\ b{:=}\top.\ b{:=}\top\ \textbf{else}\ b{:=}\top.\ b{:=}\bot\ \textbf{fi} \\ = & \textbf{if}\ \top\ \textbf{then}\ b{:=}\top\ \textbf{else}\ b{:=}\bot\ \textbf{fi} \\ = & b{:=}\top \\ = & b' \end{array}$$

But

$$\begin{array}{ll} & \textbf{if}\ b\ \textbf{then}\ R.\,P\ \textbf{else}\ R.\,Q\ \textbf{fi} \\ = & \textbf{if}\ \bot\ \textbf{then}\ b{:=}\top.\,b{:=}\top\ \textbf{else}\ b{:=}\top.\,b{:=}\bot\ \textbf{fi} \\ = & b{:=}\top.\ b{:=}\bot \\ = & \neg b' \end{array}$$

(b) $\textbf{if}\ b\ \textbf{then}\ P{\Rightarrow}Q\ \textbf{else}\ R{\Rightarrow}S\ \textbf{fi}\ \ =\ \ \textbf{if}\ b\ \textbf{then}\ P\ \textbf{else}\ R\ \textbf{fi} \Rightarrow \textbf{if}\ b\ \textbf{then}\ Q\ \textbf{else}\ S\ \textbf{fi}$

§ This is a case distributive law.

(c) $\textbf{if}\ b\ \textbf{then}\ P.\,Q\ \textbf{else}\ R.\,S\ \textbf{fi}\ \ =\ \ \textbf{if}\ b\ \textbf{then}\ P\ \textbf{else}\ R\ \textbf{fi}.\ \textbf{if}\ b\ \textbf{then}\ Q\ \textbf{else}\ S\ \textbf{fi}$

§ Let binary variable b be the only variable, let it initially be $\top$, let P be $b{:=}\bot$, Q be $b{:=}\top$, R be ok, and S be $b{:=}\bot$. Then the left side is

$$\begin{array}{ll} & \textbf{if}\ b\ \textbf{then}\ P.\,Q\ \textbf{else}\ R.\,S\ \textbf{fi} \\ = & \textbf{if}\ \top\ \textbf{then}\ b{:=}\bot.\ b{:=}\top\ \textbf{else}\ ok.\ b{:=}\bot\ \textbf{fi} \\ = & b{:=}\bot.\ b{:=}\top \\ = & b' \end{array}$$

And the right side is

$$\begin{array}{ll} & \textbf{if}\ b\ \textbf{then}\ P\ \textbf{else}\ R\ \textbf{fi}.\ \textbf{if}\ b\ \textbf{then}\ Q\ \textbf{else}\ S\ \textbf{fi} \\ = & \textbf{if}\ \top\ \textbf{then}\ b{:=}\bot\ \textbf{else}\ ok\ \textbf{fi}.\ \textbf{if}\ b\ \textbf{then}\ b{:=}\top\ \textbf{else}\ b{:=}\bot\ \textbf{fi} \\ = & b{:=}\bot.\ \textbf{if}\ b\ \textbf{then}\ b{:=}\top\ \textbf{else}\ b{:=}\bot\ \textbf{fi} \\ = & \textbf{if}\ \bot\ \textbf{then}\ b{:=}\top\ \textbf{else}\ b{:=}\bot\ \textbf{fi} \\ = & b{:=}\bot \\ = & \neg b' \end{array}$$