126 A specification is transitive if, for all states a, b, and c, if it allows the state to change from a to b, and it allows the state to change from b to c, then it allows the state to change from a to c. Prove S is transitive if and only if S is refined by $(S.S)$.

After trying the question, scroll down to the solution.

§ Let's begin with " S is refined by $S.S$ ".

$\forall \sigma, \sigma' \cdot S \Leftarrow (S.S)$ use definition of sequential composition

$= \quad \forall \sigma, \sigma' \cdot S \Leftarrow \exists \sigma'' \cdot \langle \sigma' \cdot S \rangle \sigma'' \wedge \langle \sigma \cdot S \rangle \sigma''$ antidistributive law

$= \quad \forall \sigma, \sigma' \cdot \forall \sigma'' \cdot S \Leftarrow \langle \sigma' \cdot S \rangle \sigma'' \wedge \langle \sigma \cdot S \rangle \sigma''$

reorder quantifiers (commutative law) and turn implication around (mirror law)

$= \quad \forall \sigma, \sigma'', \sigma' \cdot \langle \sigma' \cdot S \rangle \sigma'' \wedge \langle \sigma \cdot S \rangle \sigma'' \Rightarrow S$ renaming

$= \quad \forall a, b, c \cdot \langle \sigma, \sigma' \cdot S \rangle a\ b \wedge \langle \sigma, \sigma' \cdot S \rangle b\ c \Rightarrow \langle \sigma, \sigma' \cdot S \rangle a\ c$