118 Relation R is transitive if $\forall x, y, z \cdot R\ x\ y \wedge R\ y\ z \Rightarrow R\ x\ z$. Express formally that relation R is the transitive closure of relation Q (R is the strongest transitive relation that is implied by Q).

After trying the question, scroll down to the solution.

§ Here is a straightforward solution. Let $T\,R$ mean that R is a transitive relation. Formally,

$$T \;=\; \langle R\colon X\to X\to bin\cdot\ \forall x, y, z\colon X\cdot\ R\,x\,y \wedge R\,y\,z \;\Rightarrow\; R\,x\,z\rangle$$

Let $A{\ge}B$ mean that relation A is everywhere as strong as relation B. Formally,

$$A{\ge}B \;=\; \forall x, y\colon X\cdot\ A\,x\,y \Rightarrow B\,x\,y$$

Then we can say that R is the transitive closure of Q as follows.

$$T\,R \wedge Q{\ge}R \wedge \forall A\colon X\to X\to bin\cdot\ T\,A \wedge Q{\ge}A \Rightarrow R{\ge}A$$

Here is a nicer solution, but only for the special case $X = 0,..n$ for some extended natural n. Let $P\,i\,j\,k$ mean "there is a path in Q from j to k via zero or more intermediate nodes all of which are less than i". Formally,

$$P\,0 = Q$$

$$\forall i, j, k\cdot\ P\,(i{+}1)\,j\,k \;=\; P\,i\,j\,k \;\vee\; P\,i\,j\,i \wedge P\,i\,i\,k$$

Then we can say that R is the transitive closure of Q as follows:

$$R = P\,n$$

This simple definition leads to a beautiful algorithm for transitive closure.