116   Is there any harm in defining relation R with the following axioms?

$\forall x \cdot \exists y \cdot R\ x\ y$ totality

$\forall x \cdot \neg R\ x\ x$ irreflexivity

$\forall x, y, z \cdot R\ x\ y \wedge R\ y\ z \Rightarrow R\ x\ z$ transitivity

$\exists u \cdot \forall x \cdot x{=}u \vee R\ x\ u$ unity

After trying the question, scroll down to the solution.

§   ⊤   totality and unity, renaming x to z within unity

= $(\forall x\cdot \exists y\cdot R\ x\ y) \wedge (\exists u\cdot \forall z\cdot z{=}u \vee R\ z\ u)$

                                using a distributive law, move the first conjunct inside the $\exists u$

= $\exists u\cdot (\forall x\cdot \exists y\cdot R\ x\ y) \wedge (\forall z\cdot z{=}u \vee R\ z\ u)$ specialize $\forall x$ to u , splitting

⟹ $\exists u\cdot (\exists y\cdot R\ u\ y) \wedge (\forall z\cdot z{=}u \vee R\ z\ u)$

                                using a distributive law, move the second conjunct inside the $\exists y$

= $\exists u\cdot \exists y\cdot R\ u\ y \wedge (\forall z\cdot z{=}u \vee R\ z\ u)$ specialize $\forall z$ to y , splitting

⟹ $\exists u\cdot \exists y\cdot R\ u\ y \wedge (y{=}u \vee R\ y\ u)$ distributive law

= $\exists u\cdot \exists y\cdot R\ u\ y \wedge y{=}u \ \vee \ R\ u\ y \wedge R\ y\ u$ simplify and specialize the first

                        disjunct, and use transitivity on the second disjunct, and splitting (twice)

⟹ $\exists u\cdot \exists y\cdot R\ u\ u \vee R\ u\ u$ idempotence

= $\exists u\cdot R\ u\ u$ conjoin the irreflexivity axiom

= $\exists u\cdot R\ u\ u \wedge \forall x\cdot \neg R\ x\ x$ specialize $\forall x$ to u , splitting

⟹ $\exists u\cdot R\ u\ u \wedge \neg R\ u\ u$ law of noncontradiction, identity of $\exists$

=   ⊥

By proving ⊥ , we show that the given axioms are inconsistent.