One operator is the dual of another operator if it negates the result when applied to the negated operands. The zero-operand operators $\top$ and $\bot$ are each other's duals. If $\text{op}_0 \neg a \equiv \neg \text{op}_1 a$ then $\text{op}_0$ and $\text{op}_1$ are duals. If $\neg a \text{op}_0 \neg b \equiv \neg (a \text{op}_1 b)$ then $\text{op}_0$ and $\text{op}_1$ are duals. And so on for more operands.

(a) Of the 4 one-operand binary operators, there is 1 pair of duals, and 2 operators that are their own duals. Find them.

§ To answer this question, I'll use the symbols I introduce in part (f). The pair of duals is: $\top$ (always $\top$) and $\bot$ (always $\bot$). The two self-duals are: $\mathbf{I}$ (identity) and $\mathbf{+}$ (negation).

(b) Of the 16 two-operand binary operators, there are 6 pairs of duals, and 4 operators that are their own duals. Find them.

§ To answer this question, I'll use the symbols I will introduce in part (f). The six dual pairs are: $\equiv$, $\vee$, $\Rightarrow$, $\Leftarrow$, $\preceq$, $\succeq$. The four self-duals are: $<$, $>$, $\triangleright$, $\triangleleft$.

(c) What is the dual of the three-operand operator $\text{if then else } \text{fi}$? Express it using only the operator $\text{if then else } \text{fi}$.

§ Its truth table is

<table>
<thead>
<tr>
<th>$\top$</th>
<th>$\top$</th>
<th>$\bot$</th>
<th>$\bot$</th>
<th>$\bot$</th>
<th>$\bot$</th>
<th>$\bot$</th>
<th>$\bot$</th>
<th>$\bot$</th>
<th>$\bot$</th>
<th>$\bot$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\top$</td>
<td>$\bot$</td>
<td>$\top$</td>
<td>$\bot$</td>
<td>$\top$</td>
<td>$\bot$</td>
<td>$\bot$</td>
<td>$\bot$</td>
<td>$\bot$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The dual of $\text{if } a \text{ then } b \text{ else } c \text{ fi}$ is equivalent to $\text{if } a \text{ then } c \text{ else } b \text{ fi}$.

(d) The dual of a binary expression without variables is formed as follows: replace each operator with its dual, adding parentheses if necessary to maintain the precedence. Explain why the dual of a theorem is an antitheorem, and vice versa.

§ I will show that for every expression $P$ without variables, $(\text{dual of } P) = \neg P$. I do so by induction on the structure of expression $P$. The two binary values give us two base cases.

\[(\text{dual of } \top) = \bot \quad \text{use the dual-forming rules}\]
\[(\text{dual of } \bot) = \neg \top \quad \text{use the dual-forming rules}\]

There is an induction step for each of the binary operators. Suppose (this is an induction hypothesis) that $(\text{dual of } P) = \neg P$. Then

\[(\text{dual of } \neg P) = \neg(\text{dual of } P) \quad \text{use the dual-forming rules}\]
\[= \neg \neg P \quad \text{use the induction hypothesis}\]
\[= P \quad \text{use duality law}\]

Suppose that $(\text{dual of } P) = \neg P$ and $(\text{dual of } Q) = \neg Q$. Then

\[(\text{dual of } P \land Q) = (\text{dual of } P) \lor (\text{dual of } Q) \quad \text{use the duality laws}\]
\[= \neg P \lor \neg Q \quad \text{use duality law}\]
\[= \neg(P \land Q) \quad \text{use duality law}\]

And similarly for all other operators.

(e) Let $P$ be a binary expression without variables. From part (d) we know that every binary expression without variables of the form $(\text{dual of } P) = \neg P$ is a theorem. Therefore, to find the dual of a binary expression with variables, we must
replace each operator by its dual and negate each variable. For example, if $a$ and $b$ are binary variables, then the dual of $a \land b$ is $\neg a \lor \neg b$. And since

\[(\text{dual of } a \land b) \equiv \neg(a \land b)\]

we have one of the Duality Laws:

\[\neg a \lor \neg b \equiv \neg(a \land b)\]

The other of the Duality Laws is obtained by equating the dual and negation of $a \lor b$.

Obtain five laws that do not appear in this book by equating a dual with a negation.

§ From $a=b$ we get $\neg a \lor \neg b \equiv \neg(a=b)$

From if $a$ then $b$ else $c$ fi we get if $\neg a$ then $\neg c$ else $\neg b$ fi $\equiv \neg$if $a$ then $b$ else $c$ fi

From $a=b \land c$ we get $\neg a \lor \neg b \lor \neg c \equiv \neg(a\land b \land c)$

From $a=b \lor c$ we get $\neg a \lor \neg b \land \neg c \equiv \neg(a\lor b \land c)$

From $a = (b\land c)$ we get $\neg a \lor (\neg b \lor \neg c) \equiv \neg(a = (b\land c))$

(f) Dual operators have truth tables that are each other's vertical mirror reflections. For example, the truth table for $\land$ (below left) is the vertical mirror reflection of the truth table for $\lor$ (below right).

\[
\begin{array}{c|c|c}
\land & T & T \\
\hline
T & T & T \\
\hline
\bot & \bot & \bot \\
\end{array}
\quad
\begin{array}{c|c|c}
\lor & T & T \\
\hline
T & T & T \\
\hline
\bot & \bot & \bot \\
\end{array}
\]

Design symbols (you may redesign existing symbols where necessary) for the 4 one-operand and 16 two-operand binary operators according to the following criteria.

(i) Dual operators should have symbols that are vertical mirror reflections (like $\land$ and $\lor$). This implies that self-dual operators have vertically symmetric symbols, and all others have vertically asymmetric symbols.

(ii) If $a \ op_0 b \equiv b \ op_1 a$ then $\ op_0$ and $\ op_1$ should have symbols that are horizontal mirror reflections (like $\Rightarrow$ and $\Leftarrow$). This implies that symmetric operators have horizontally symmetric symbols, and all others have horizontally asymmetric symbols.