Let $f$ and $g$ be functions from $\text{nat}$ to $\text{nat}$.

(a) For what $f$ do we have the theorem $fg = g$?
(b) For what $f$ do we have the theorem $gf = g$?

After trying the question, scroll down to the solution.
(a) For what $f$ do we have the theorem $f \circ g = g$?
§ Equality $f \circ g = g$ means, first, that the domains are equal.

\[
\begin{align*}
\square (f \circ g) &= \square g \\
\equiv & \quad (\$x: \square g \cdot g \ x: \square f) = \square g \\
\equiv & \quad (\$x: \text{nat} \cdot g \ x: \text{nat}) = \text{nat} \\
\equiv & \quad (\$x: \text{nat} \cdot \top) = \text{nat} \\
\equiv & \quad \text{nat} = \text{nat} \\
\equiv & \quad \top
\end{align*}
\]
so that's no constraint. Equality also means that the results are equal.

\[
\begin{align*}
\forall x : \text{nat} \cdot (f \circ g) \ x &= g \ x \\
\equiv & \quad \forall x : \text{nat} \cdot f (g \ x) = g \ x
\end{align*}
\]
So $f$ must be the identity function on the range of $g$.

\[
\begin{align*}
\forall x : \text{nat} \cdot f x &= x
\end{align*}
\]

(b) For what $f$ do we have the theorem $g \circ f = g$?
§ The domains of $g \circ f$ and $g$ must be equal, and they are both $\text{nat}$. The results must also be equal.

\[
\begin{align*}
\forall x : \text{nat} \cdot (g \circ f) \ x &= g \ x \\
\equiv & \quad \forall x : \text{nat} \cdot g (f \ x) = g \ x
\end{align*}
\]
For any $x$ such that $f x \neq x$, $g$ must give the same result for both $f x$ and $x$. If $f$ is the identity function, then $g \circ f = g$. If $g$ is a constant function, then $g \circ f = g$. 