106  (Russell's paradox) Define \( rus = \langle f: \text{null} \to \text{bin} \cdot \neg f f \rangle \).

(a) Can we prove \( rus \ rus = \neg rus \ rus \)?

§ To apply \( rus \) to \( rus \), we must first prove that \( rus \) is in the domain of \( rus \)

\[
\begin{align*}
\text{rus}: \Box \text{rus} & \quad \text{use Domain Axiom} \\
\equiv & \quad \text{rus}: \text{null} \to \text{bin} \\
\equiv & \quad \text{null}: \Box \text{rus} \land \forall x: \text{null} \cdot \text{rus} x: \text{bin} \quad \text{Both conjuncts are instances of axioms} \\
\equiv & \quad \top
\end{align*}
\]

(so it is) and that \( rus \) is elementary (it isn't) or that it occurs only once and in a

distributing context (it occurs twice). So we cannot use the application law to apply \( rus \)
to \( rus \). But let's do it anyway just to see what we get.

\[
\begin{align*}
\text{rus rus} & \quad \text{replace first \( rus \) by its equal} \\
\equiv & \quad \langle f: \text{null} \to \text{bin} \cdot \neg f f \rangle \text{rus} \quad \text{use Application Law (this step is wrong)} \\
\equiv & \quad \neg \text{rus rus}
\end{align*}
\]

(b) Is this an inconsistency?

§ Since we could not use the Application Law to apply \( rus \) to \( rus \), we don't have a

problem. But even if we could, we still wouldn't have a problem. In Exercise 27 we saw
two instances of \( A = \neg A \), namely, \( \text{null} = \neg \text{null} \) and \( \text{bin} = \neg \text{bin} \). Now we would have

one more instance. We would have inconsistency if we had an elementary binary

expression that is both a theorem and an antitheorem. The expressions \( rus \) and \( rus \ rus \)

are not elementary. The expression \( rus \ rus = \neg rus \ rus \) is elementary, but we cannot use

the Completion Rule to prove it is an antitheorem because \( rus \ rus \) is not elementary. So

we do not have an inconsistency.

(c) Can we add the axiom \( \neg f: \Box f \)? Would it help?

§ Since we don't have an inconsistency, we aren't in trouble, and we don't need help. If we
did have an inconsistency, it never helps to add an axiom. Adding axioms can only add

theorems and antitheorems, and we would want to decrease the quantity of theorems and/
or antitheorems. We could take away the axiom defining \( rus \), but that wouldn't help

either because (since it isn't a recursive definition) we could always use

\( \langle f: \text{null} \to \text{bin} \cdot \neg f f \rangle \)

in place of \( rus \). Since we have an instance of \( f: \Box f \), namely \( rus: \Boxrus \), adding

\( \neg f: \Box f \)

as an axiom would cause inconsistency.