106 (Russell's paradox) Define $rus = \langle f: null \rightarrow bin \cdot \neg f\, f \rangle$.

(a) Can we prove $rus\ rus = \neg\ rus\ rus$?

(b) Is this an inconsistency?

(c) Can we add the axiom $\neg\ f: \square f$? Would it help?

After trying the question, scroll down to the solution.

(a) Can we prove $rus\ rus\ =\ \neg\ rus\ rus$?

§ To apply rus to rus , we must first prove that rus is in the domain of rus

$rus: \Box rus$ — use Domain Axiom

$=\ rus: null \rightarrow bin$ — use Function Inclusion Axiom

$=\ null: \Box rus\ \land\ \forall x: null \cdot\ rus\ x: bin$ — Both conjuncts are instances of axioms

$=\ \top$

(so it is) and that rus is elementary (it isn't) or that it occurs only once and in a distributing context (it occurs twice). So we cannot use the application law to apply rus to rus . But let's do it anyway just to see what we get.

$rus\ rus$ — replace first rus by its equal

$=\ \langle f: null \rightarrow bin \cdot\ \neg\ f\,f \rangle\ rus$ — use Application Law (this step is wrong)

$=\ \neg\ rus\ rus$

(b) Is this an inconsistency?

§ Since we could not use the Application Law to apply rus to rus , we don't have a problem. But even if we could, we still wouldn't have a problem. In Exercise 27 we saw two instances of $A = \neg A$, namely, $null = \neg null$ and $bin = \neg bin$. Now we would have one more instance. We would have inconsistency if we had an <u>elementary</u> binary expression that is both a theorem and an antitheorem. The expressions rus and $rus\ rus$ are <u>not</u> elementary. The expression $rus\ rus = \neg\ rus\ rus$ <u>is</u> elementary, but we cannot use the Completion Rule to prove it is an antitheorem because $rus\ rus$ is not elementary. So we do not have an inconsistency.

(c) Can we add the axiom $\neg\ f: \Box f$? Would it help?

§ Since we don't have an inconsistency, we aren't in trouble, and we don't need help. If we did have an inconsistency, it <u>never</u> helps to add an axiom. Adding axioms can only add theorems and antitheorems, and we would want to decrease the quantity of theorems and/or antitheorems. We could take away the axiom defining rus , but that wouldn't help either because (since it isn't a recursive definition) we could always use

$\langle f: null \rightarrow bin \cdot\ \neg\ f\,f \rangle$

in place of rus . Since we have an instance of $f: \Box f$, namely $rus: \Box rus$, adding

$\neg\ f: \Box f$

as an axiom would cause inconsistency.