The following statements are made.

- All unicorns are white.
- All unicorns are black.
- No unicorn is both white and black.

Are these statements consistent? What, if anything, can we conclude about unicorns?

After trying the question, scroll down to the solution.
Let $\text{unicorn}$ be all unicorns. Let $\text{white}$ and $\text{black}$ be predicates on unicorns. Then

All unicorns are white:
(a) $\forall u: \text{unicorn} \cdot \text{white } u$

All unicorns are black:
(b) $\forall u: \text{unicorn} \cdot \text{black } u$

No unicorn is both white and black:
(c) $\neg \exists u: \text{unicorn} \cdot \text{white } u \land \text{black } u$

Suppose we take (a), (b), and (c) as axioms.

\[
\top \quad \text{(a), (b), and (c) are axioms)
\]

\[
= (\forall u: \text{unicorn} \cdot \text{white } u) \land (\forall u: \text{unicorn} \cdot \text{black } u) \land (\neg \exists u: \text{unicorn} \cdot \text{white } u \land \text{black } u)
\]

Using a duality law (deMorgan) on (c), we can change it to a universal quantification:

\[
= (\forall u: \text{unicorn} \cdot \text{white } u) \land (\forall u: \text{unicorn} \cdot \text{black } u) \land (\forall u: \text{unicorn} \cdot \neg (\text{white } u \land \text{black } u))
\]

Now we can use a splitting law to combine the three main conjuncts

\[
= \forall u: \text{unicorn} \cdot (\text{white } u \land \text{black } u) \land \neg (\text{white } u \land \text{black } u) \text{ Law of Noncontradiction}
\]

\[
= \forall u: \text{unicorn} \cdot \bot \quad \text{one-case}
\]

\[
= \text{if unicorn=\text{null} then } \forall u: \text{unicorn} \cdot \bot \text{ else } \forall u: \text{unicorn} \cdot \bot \text{ fi}
\]

In then-part, use if-part as context, and quantifier law $\forall v: \text{null} \cdot b$. In else-part, use negation of if-part as context, and idempotent law.

\[
= \text{if unicorn=\text{null} then } \top \text{ else } \bot \text{ fi}
\]

there ought to be a law

\[
= \text{unicorn=\text{null}}
\]

If we are given (a), (b), and (c) as axioms, we must conclude that there are no unicorns.