100 Prove that the square of an odd natural number is odd, and the square of an even natural number is even.

After trying the question, scroll down to the solution.

§ The even naturals are $2×nat$ and the odd naturals are $2×nat + 1$. To say that the square of an odd natural is odd is easy:

$$(2×nat + 1)^2: 2×nat + 1$$

But arithmetic on bunches is tricky; for example, nat^2 and $nat×nat$ differ; $2×nat$ and $nat+nat$ differ. To be safe, we prove

$$
\begin{array}{lll}
 & \forall a: 2×nat + 1· \exists b: 2×nat + 1· a^2 = b & \text{change of variable, twice} \\
= & \forall n: nat· \exists m: nat· (2×n + 1)^2 = 2×m + 1 & \text{various number laws} \\
= & \forall n: nat· \exists m: nat· 2×(2×n^2 + 2×n) + 1 = 2×m + 1 & \text{generalization} \\
\Leftarrow & \forall n: nat· 2×(2×n^2 + 2×n) + 1 = 2×(2×n^2 + 2×n) + 1 & \text{reflexivity and identity laws} \\
= & \top &
\end{array}
$$

To prove the square of an even natural number is even, we prove

$$
\begin{array}{lll}
 & \forall a: 2×nat· \exists b: 2×nat· a^2 = b & \text{change of variable, twice} \\
= & \forall n: nat· \exists m: nat· (2×n)^2 = 2×m & \text{various number laws} \\
= & \forall n: nat· \exists m: nat· 2×(2×n^2) = 2×m & \text{generalization} \\
\Leftarrow & \forall n: nat· 2×(2×n^2) = 2×(2×n^2) & \text{reflexivity and identity laws} \\
= & \top &
\end{array}
$$