Consider a fully parenthesized expression containing only the symbols $\top$, $\bot$, $=$, $\vdash$ ( ) in any quantity and any syntactically acceptable order.

§ The proofs will be by induction over the structure of the expressions. Every fully parenthesized expression containing only the symbols $\top$, $\bot$, $=$, $\vdash$ ( ) has one of the following four forms: $\top$, $\bot$, $(a=b)$, $(a\vdash b)$, where $a$ and $b$ are fully parenthesized expression containing only the symbols $\top$, $\bot$, $=$, $\vdash$ ( ).

(a) Show that all syntactically acceptable rearrangements are equivalent.

There are four alternatives. The first two alternatives are just a single symbol, so there are no rearrangements, so all zero rearrangements are equivalent. That's the base case. Now for the induction step.

Suppose the expression is $(a=b)$ for some expressions $a$ and $b$. The ways of rearranging $(a=b)$ are:

(a) rearrange $a$
(b) rearrange $b$
(c) change $(a=b)$ to $(b=a)$

First, consider (a). Make the inductive hypothesis that rearranging $a$ results in an expression that is equivalent to $a$. Then any expression with subexpression $a$ is equivalent to the same expression with subexpression $a$ replaced by its rearrangement. (This is formalized as the generic law of transparency.) Similarly for (b). For (c), we have the generic law of symmetry of $=$. That completes the proof for expressions of the form $(a=b)$.

Finally, suppose the expression is $(a\vdash b)$ for some expressions $a$ and $b$. The ways of rearranging $(a\vdash b)$ are:

(a) rearrange $a$
(b) rearrange $b$
(c) change $(a\vdash b)$ to $(b\vdash a)$

First, consider (a). Make the inductive hypothesis that rearranging $a$ results in an expression that is equivalent to $a$. Then any expression with subexpression $a$ is equivalent to the same expression with subexpression $a$ replaced by its rearrangement. (This is formalized as the generic law of transparency.) Similarly for (b). For (c), we have the generic law of symmetry of $\vdash$.

That completes the proof.

(b) Show that it is equivalent to any expression obtained from it by making an even number of the following substitutions: $\top$ for $\bot$, $\bot$ for $\top$, $=$ for $\vdash$, $\vdash$ for $=$.

§ Zero substitutions means the same expression, which is obviously equivalent. I will show that by making a single one of those substitutions, the expression is negated. Therefore two substitutions are a double negation, which is an equivalent expression. And so on for more substitutions.

If the expression is $\top$, the only substitution is $\bot$ for $\top$, and $\bot$ is the negation of $\top$.

If the expression is $\bot$, the only substitution is $\top$ for $\bot$, and $\top$ is the negation of $\bot$. 
Suppose the expression is \((a=b)\) for some expressions \(a\) and \(b\). The ways of making one substitution in \((a=b)\) are:

(a) make one substitution in \(a\)
(b) make one substitution in \(b\)
(c) change \((a=b)\) to \((a\neq b)\)

First, consider (a). Make the inductive hypothesis that one substitutions in \(a\) negates \(a\), resulting in an expression equivalent to \(\neg(a=b)\).

\[
\begin{align*}
\neg(a=b) & \quad \text{exclusion} \\
\equiv & \quad (a\neq b) \\
\equiv & \quad \neg(a=b)
\end{align*}
\]

so making one substitution in \(a\) negates \((a=b)\). Similarly for (b). For (c),

\[
\begin{align*}
(a\neq b) & \quad \text{generic inequality} \\
\equiv & \quad \neg\neg(a=b) \\
\equiv & \quad \neg(a\neq b)
\end{align*}
\]

That completes the proof for expressions of the form \((a=b)\). Finally, suppose the expression is \((a\neq b)\) for some expressions \(a\) and \(b\). The ways of making one substitution in \((a\neq b)\) are:

(a) make one substitution in \(a\)
(b) make one substitution in \(b\)
(c) change \((a\neq b)\) to \((a=b)\)

First, consider (a). Make the inductive hypothesis that one substitutions in \(a\) negates \(a\), resulting in an expression equivalent to \(\neg(a\neq b)\).

\[
\begin{align*}
\neg(a\neq b) & \quad \text{generic inequality} \\
\equiv & \quad \neg\neg(a\neq b) \\
\equiv & \quad \neg(a\neq b)
\end{align*}
\]

so making one substitution in \(a\) negates \((a\neq b)\). Similarly for (b). For (c),

\[
\begin{align*}
(a=b) & \quad \text{double negation} \\
\equiv & \quad \neg
\neg(a=b) \\
\equiv & \quad \neg(a\neq b)
\end{align*}
\]

That completes the proof