Quantifier Use

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To illustrate how we use quantifiers in formalization problems, here is the solution to Q.72(a) of the textbook. We are asked to formalize:

> natural n is the largest proper (neither 1 nor m) factor of natural m

This is a sentence that talks about natural numbers n and m. Therefore, these are the only free (not bound by quantifiers) variables that we are allowed to use in the formal expression.

Formalization usually needs a lot of *rephrasing* steps. The purpose of rephrasing is to bring the informal sentence closer to the formal notation. We do that by making the informal sentence *more precise*.

In our problem, the first step is to rephrase the sentence so that the logical connectives in it become apparent:

> n is not m and n is not 1 and n is a factor of m and n is greater than or equal to any other natural number that has these properties

Formalization of most of this sentence is immediate:

$n \neq m \;\wedge\; n \neq 1 \;\wedge\;$ (n is a factor of m)
$\wedge\;$ (n is greater than or equal to any other natural number that has these properties)

We can further simplify as follows:

$\neg n : 1, m \;\wedge\;$ (n is a factor of m)
$\wedge\;$ (n is greater than or equal to any other natural number that has these properties)

We are closer to the formal language. Let us focus on the last conjunct, which is still informal:

> n is greater than or equal to any other natural number that has these properties

How do we formalize "these properties"? We have to be more specific:

> n is greater than or equal to any other natural number that is not m and is not 1 and is a factor of m

To formalize "n is greater than or equal to any natural number that has property p"? We can rephrase by saying "all natural numbers that have property p are less than or equal to n". Or, "for any natural number x, if x has property p, then x is less than or equal to n". So the last conjunct is formalized as follows:

$$\forall x : nat \cdot \ \neg x : 1, m \ \wedge \ (x \text{ is a factor of } m) \ \ \Rightarrow \ x \leq n$$

Now, we know that a factor of m cannot be greater than m itself and it cannot be 0. So, the conjunct (x is a factor of m) allows us to restrict the domain of our formalization:

$$\forall x : 1,..m+1 \cdot \ \neg x : 1, m \ \wedge \ (x \text{ is a factor of } m) \ \ \Rightarrow \ x \leq n$$

Also, the conjunct $x : 1, m$ excludes values 1 and m from the domain:

$$\forall x : 2,..m \cdot \ (x \text{ is a factor of } m) \ \ \Rightarrow \ x \leq n$$

Finally, we must formalize the part "is a factor", which remains informal. We find this part in two different places in our formalization. Furthermore, in these two places, we talk about different numbers. Finally, the formalization of "is a factor" is by itself non-trivial. For all these reasons, we will define "is a factor" as a separate *predicate* of two variables:

$$factor = \langle x : nat \rightarrow \langle y : nat \rightarrow x \text{ is a factor of } y \rangle \rangle$$

The formalization of the original sentence is complete if we use that predicate:

$$\neg n : 1, m \ \wedge \ factor \ n \ m \ \wedge \ \forall x : 2,..m \cdot \ factor \ x \ m \ \ \Rightarrow \ x \leq n$$

The *factor* predicate is formalized as follows: x is a factor of y means:

there is a natural number z such that $y = x \times z$

Formally:

$$factor = \langle x : nat \rightarrow \langle y : nat \rightarrow \exists z : nat \cdot \ y = x \times z \rangle \rangle$$