a Practical Theory of Programming

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11.4 Laws

11.4.0 Binary

Let a , b , c , d , and e be binary.

Binary Laws

$\top$

$\neg\bot$

Law of Excluded Middle (Tertium non Datur)

$a \lor \neg a$

Law of Noncontradiction

$\neg(a \land \neg a)$

Base Laws

$\neg(a \land \bot)$

$a \lor \top$

$a \Rightarrow \top$

$\bot \Rightarrow a$

Identity Laws

$\top \land a \;=\; a$

$\bot \lor a \;=\; a$

$\top \Rightarrow a \;=\; a$

$\top = a \;=\; a$

Idempotent Laws

$a \land a \;=\; a$

$a \lor a \;=\; a$

Reflexive Laws

$a \Rightarrow a$

$a = a$

Laws of Indirect Proof

$\neg a \Rightarrow \bot \;=\; a$ (Reductio ad Absurdum)

$\neg a \Rightarrow a \;=\; a$

Law of Specialization

$a \land b \Rightarrow a$

Associative Laws

$a \land (b \land c) \;=\; (a \land b) \land c$

$a \lor (b \lor c) \;=\; (a \lor b) \lor c$

$a = (b = c) \;=\; (a = b) = c$

$a \neq (b \neq c) \;=\; (a \neq b) \neq c$

$a = (b \neq c) \;=\; (a = b) \neq c$

Mirror Law

$a \Leftarrow b \;=\; b \Rightarrow a$

Law of Double Negation

$\neg\neg a \;=\; a$

Duality Laws (deMorgan)

$\neg(a \land b) \;=\; \neg a \lor \neg b$

$\neg(a \lor b) \;=\; \neg a \land \neg b$

Laws of Exclusion

$a \Rightarrow \neg b \;=\; b \Rightarrow \neg a$

$a = \neg b \;=\; a \neq b \;=\; \neg a = b$

Laws of Inclusion

$a \Rightarrow b \;=\; \neg a \lor b$ (Material Implication)

$a \Rightarrow b \;=\; (a \land b \;=\; a)$

$a \Rightarrow b \;=\; (a \lor b \;=\; b)$

Absorption Laws

$a \land (a \lor b) \;=\; a$

$a \lor (a \land b) \;=\; a$

Laws of Direct Proof

$(a \Rightarrow b) \land a \Rightarrow b$ (Modus Ponens)

$(a \Rightarrow b) \land \neg b \Rightarrow \neg a$ (Modus Tollens)

$(a \lor b) \land \neg a \Rightarrow b$(Disjunctive Syllogism)

Transitive Laws

$(a \land b) \land (b \land c) \Rightarrow (a \land c)$

$(a \Rightarrow b) \land (b \Rightarrow c) \Rightarrow (a \Rightarrow c)$

$(a = b) \land (b = c) \Rightarrow (a = c)$

$(a \Rightarrow b) \land (b = c) \Rightarrow (a \Rightarrow c)$

$(a = b) \land (b \Rightarrow c) \Rightarrow (a \Rightarrow c)$

Distributive Laws (Factoring)

$a \land (b \land c) \;=\; (a \land b) \land (a \land c)$

$a \land (b \lor c) \;=\; (a \land b) \lor (a \land c)$

$a \lor (b \land c) \;=\; (a \lor b) \land (a \lor c)$

$a \lor (b \lor c) \;=\; (a \lor b) \lor (a \lor c)$

$a \lor (b \Rightarrow c) \;=\; (a \lor b) \Rightarrow (a \lor c)$

$a \lor (b = c) \;=\; (a \lor b) = (a \lor c)$

$a \Rightarrow (b \land c) \;=\; (a \Rightarrow b) \land (a \Rightarrow c)$

$a \Rightarrow (b \lor c) \;=\; (a \Rightarrow b) \lor (a \Rightarrow c)$

$a \Rightarrow (b \Rightarrow c) \;=\; (a \Rightarrow b) \Rightarrow (a \Rightarrow c)$

$a \Rightarrow (b = c) \;=\; (a \Rightarrow b) = (a \Rightarrow c)$

Symmetry Laws (Commutative Laws)

$$a \wedge b \quad = \quad b \wedge a$$
$$a \vee b \quad = \quad b \vee a$$
$$a = b \quad = \quad b = a$$
$$a \neq b \quad = \quad b \neq a$$

Antisymmetry Law (Double Implication)

$$(a \Rightarrow b) \wedge (b \Rightarrow a) \quad = \quad a = b$$

Laws of Discharge

$$a \wedge (a \Rightarrow b) \quad = \quad a \wedge b$$
$$a \Rightarrow (a \wedge b) \quad = \quad a \Rightarrow b$$

Antimonotonic Law

$$a \Rightarrow b \quad \Longrightarrow \quad (b \Rightarrow c) \Rightarrow (a \Rightarrow c)$$

Monotonic Laws

$$a \Rightarrow b \quad \Longrightarrow \quad c \wedge a \Rightarrow c \wedge b$$
$$a \Rightarrow b \quad \Longrightarrow \quad c \vee a \Rightarrow c \vee b$$
$$a \Rightarrow b \quad \Longrightarrow \quad (c \Rightarrow a) \Rightarrow (c \Rightarrow b)$$

Law of Generalization

$$a \Rightarrow a \vee b$$

Antidistributive Laws

$$a \wedge b \Rightarrow c \quad = \quad (a \Rightarrow c) \vee (b \Rightarrow c)$$
$$a \vee b \Rightarrow c \quad = \quad (a \Rightarrow c) \wedge (b \Rightarrow c)$$

Laws of Portation

$$a \wedge b \Rightarrow c \quad = \quad a \Rightarrow (b \Rightarrow c)$$
$$a \wedge b \Rightarrow c \quad = \quad a \Rightarrow \neg b \vee c$$

Laws of Conflation

$$(a \Rightarrow b) \wedge (c \Rightarrow d) \quad \Longrightarrow \quad a \wedge c \Rightarrow b \wedge d$$
$$(a \Rightarrow b) \wedge (c \Rightarrow d) \quad \Longrightarrow \quad a \vee c \Rightarrow b \vee d$$

Contrapositive Law

$$a \Rightarrow b \quad = \quad \neg b \Rightarrow \neg a$$

Laws of Equality and Difference

$$a{=}b \quad = \quad (a \wedge b) \vee (\neg a \wedge \neg b)$$
$$a{\neq}b \quad = \quad (a \wedge \neg b) \vee (\neg a \wedge b)$$

Law of Resolution

$$a \wedge c \quad \Longrightarrow \quad (a \vee b) \wedge (\neg b \vee c) \quad = \quad (a \wedge \neg b) \vee (b \wedge c) \quad \Longrightarrow \quad a \vee c$$

Case Creation Laws

$$a \;=\; \textbf{if } b \textbf{ then } b{\Rightarrow}a \textbf{ else } \neg b \Rightarrow a \textbf{ fi}$$
$$a \;=\; \textbf{if } b \textbf{ then } b{\wedge}a \textbf{ else } \neg b \wedge a \textbf{ fi}$$
$$a \;=\; \textbf{if } b \textbf{ then } b{=}a \textbf{ else } b{\neq}a \textbf{ fi}$$

Case Absorption Laws

$$\textbf{if } a \textbf{ then } b \textbf{ else } c \textbf{ fi} \;=\; \textbf{if } a \textbf{ then } a{\wedge}b \textbf{ else } c \textbf{ fi}$$
$$\textbf{if } a \textbf{ then } b \textbf{ else } c \textbf{ fi} \;=\; \textbf{if } a \textbf{ then } a{\Rightarrow}b \textbf{ else } c \textbf{ fi}$$
$$\textbf{if } a \textbf{ then } b \textbf{ else } c \textbf{ fi} \;=\; \textbf{if } a \textbf{ then } a{=}b \textbf{ else } c \textbf{ fi}$$
$$\textbf{if } a \textbf{ then } b \textbf{ else } c \textbf{ fi} \;=\; \textbf{if } a \textbf{ then } b \textbf{ else } \neg a \wedge c \textbf{ fi}$$
$$\textbf{if } a \textbf{ then } b \textbf{ else } c \textbf{ fi} \;=\; \textbf{if } a \textbf{ then } b \textbf{ else } a{\vee}c \textbf{ fi}$$
$$\textbf{if } a \textbf{ then } b \textbf{ else } c \textbf{ fi} \;=\; \textbf{if } a \textbf{ then } b \textbf{ else } a{\neq}c \textbf{ fi}$$

Case Analysis Laws

$$\textbf{if } a \textbf{ then } b \textbf{ else } c \textbf{ fi} \quad = \quad (a \wedge b) \vee (\neg a \wedge c)$$
$$\textbf{if } a \textbf{ then } b \textbf{ else } c \textbf{ fi} \quad = \quad (a{\Rightarrow}b) \wedge (\neg a \Rightarrow c)$$

One Case Laws

$$\textbf{if } a \textbf{ then } \top \textbf{ else } b \textbf{ fi} \quad = \quad a{\vee}b$$
$$\textbf{if } a \textbf{ then } \bot \textbf{ else } b \textbf{ fi} \quad = \quad \neg a \wedge b$$
$$\textbf{if } a \textbf{ then } b \textbf{ else } \top \textbf{ fi} \quad = \quad a{\Rightarrow}b$$
$$\textbf{if } a \textbf{ then } b \textbf{ else } \bot \textbf{ fi} \quad = \quad a{\wedge}b$$

Case Distributive Laws (Case Factoring)

$$\neg\, \textbf{if } a \textbf{ then } b \textbf{ else } c \textbf{ fi} \quad = \quad \textbf{if } a \textbf{ then } \neg b \textbf{ else } \neg c \textbf{ fi}$$
$$\textbf{if } a \textbf{ then } b \textbf{ else } c \textbf{ fi} \wedge d \quad = \quad \textbf{if } a \textbf{ then } b{\wedge}d \textbf{ else } c{\wedge}d \textbf{ fi}$$

and similarly replacing $\wedge$ by any of $\vee \;=\; \neq \;\Rightarrow\; \Leftarrow$

$$\textbf{if } a \textbf{ then } b \wedge c \textbf{ else } d \wedge e \textbf{ fi} \quad = \quad \textbf{if } a \textbf{ then } b \textbf{ else } d \textbf{ fi} \wedge \textbf{if } a \textbf{ then } c \textbf{ else } e \textbf{ fi}$$

and similarly replacing $\wedge$ by any of $\vee \;=\; \neq \;\Rightarrow\; \Leftarrow$

End of Binary

11.4.1 Generic

The operators $=\ \neq$ **if then else fi** apply to every type of expression (but the first operand of **if then else fi** must be binary), with the laws

$x = x$	reflexivity	**if** $\top$ **then** x **else** y **fi** $= x$	case base
$x{=}y \;=\; y{=}x$	symmetry	**if** $\bot$ **then** x **else** y **fi** $= y$	case base
$x{=}y \land y{=}z \;\Rightarrow\; x{=}z$	transitivity	**if** a **then** x **else** x **fi** $= x$	case idempotent
$x{=}y \;\Rightarrow\; f\,x = f\,y$	transparency	**if** a **then** x **else** y **fi** $=$ **if** $\neg a$ **then** y **else** x **fi**	
$x{\neq}y \;=\; \neg(x{=}y)$	unequality		case reversal

The operators $< \le > \ge$ apply to numbers, characters, strings, and lists, with the laws

$x{\le}x$	reflexivity
$\neg\, x{<}x$	irreflexivity
$\neg(x{<}y \land x{>}y)$	exclusivity
$\neg(x{<}y \land x{=}y)$	exclusivity
$x{\le}y \land y{\le}x = x{=}y$	antisymmetry
$x{\le}y \land y{\le}z \Rightarrow x{\le}z$	transitivity
$x{<}y \land y{<}z \Rightarrow x{<}z$	transitivity
$x{\le}y \;=\; x{<}y \lor x{=}y$	inclusivity
$x{>}y \;=\; y{<}x$	mirror
$x{\ge}y \;=\; y{\le}x$	mirror
$x{<}y \lor x{=}y \lor x{>}y$	totality, trichotomy

End of Generic

11.4.2 Numbers

Let d be a sequence of (zero or more) digits, and let x , y , and z be numbers.

$d0{+}1 = d1$	counting
$d1{+}1 = d2$	counting
$d2{+}1 = d3$	counting
$d3{+}1 = d4$	counting
$d4{+}1 = d5$	counting
$d5{+}1 = d6$	counting
$d6{+}1 = d7$	counting
$d7{+}1 = d8$	counting
$d8{+}1 = d9$	counting
$d9{+}1 = (d{+}1)0$	counting (see Exercise 32)
$x{+}0 = x$	identity
$x{+}y = y{+}x$	symmetry
$x{+}(y{+}z) \;=\; (x{+}y){+}z$	associativity
$-\infty{<}x{<}\infty \Rightarrow (x{+}y = x{+}z \;=\; y{=}z)$	cancellation
$-\infty{<}x \Rightarrow\; \infty{+}x = \infty$	absorption
$x{<}\infty \;\Rightarrow\; -\infty + x = -\infty$	absorption
$-x \;=\; 0 - x$	negation
$- -x = x$	self-inverse
$-(x{+}y) = -x + -y$	distributivity
$-(x{-}y) = y{-}x$	antisymmetry
$-(x{\times}y) = -x \times y$	semi-distributivity
$-(x/y) = -x / y$	semi-distributivity
$x{-}0 = x$	identity

$x-y \;=\; x+-y$	subtraction
$x+(y-z) \;=\; (x+y)-z$	associativity
$-\infty<x<\infty \;\Rightarrow\; (x-y=x-z \;\;=\;\; y=z)$	cancellation
$-\infty<x<\infty \;\Rightarrow\; x-x=0$	inverse
$x<\infty \;\Rightarrow\; \infty-x=\infty$	absorption
$-\infty<x \;\Rightarrow\; -\infty-x=-\infty$	absorption
$-\infty<x<\infty \;\Rightarrow\; x\times 0=0$	base
$x\times 1=x$	identity
$x\times y=y\times x$	symmetry
$x\times(y+z)=x\times y+x\times z$	distributivity
$x\times(y\times z)=(x\times y)\times z$	associativity
$-\infty<x<\infty \wedge x\ne 0 \;\Rightarrow\; (x\times y=x\times z \;\;=\;\; y=z)$	cancellation
$0<x \;\Rightarrow\; x\times\infty=\infty$	absorption
$0<x \;\Rightarrow\; x\times-\infty \;=\; -\infty$	absorption
$x/1=x$	identity
$-\infty<x<\infty \wedge x\ne 0 \;\Rightarrow\; x/x=1$	inverse
$x\times(y/z)=(x\times y)/z=x/(z/y)$	multiplication-division
$y\ne 0 \;\Rightarrow\; (x/y)/z=x/(y\times z)$	multiplication-division
$-\infty<x<\infty \;\Rightarrow\; x/\infty=0=x/-\infty$	annihilation
$-\infty<x<\infty \;\Rightarrow\; x^0=1$	base
$x^1=x$	identity
$x^{y+z}=x^y\times x^z$	exponents
$x^{y\times z}=(x^y)^z$	exponents
$-\infty<0<1<\infty$	direction
$x<y \;\;=\;\; -y<-x$	reflection
$-\infty<x<\infty \;\Rightarrow\; (x+y<x+z \;\;=\;\; y<z)$	cancellation, translation
$0<x<\infty \;\Rightarrow\; (x\times y<x\times z \;\;=\;\; y<z)$	cancellation, scale
$x<y \vee x=y \vee x>y$	trichotomy
$-\infty \le x \le \infty$	extremes

End of Numbers

11.4.3 Bunches

Let x and y be elements (binary values, numbers, characters, sets, strings and lists of elements).

$x\colon y \;\;=\;\; x=y$	elementary law
$x\colon A,B \;\;=\;\; x\colon A \;\vee\; x\colon B$	compound law
$A,A=A$	idempotence
$A,B=B,A$	symmetry
$A,(B,C)=(A,B),C$	associativity
$A‘A=A$	idempotence
$A‘B=B‘A$	symmetry
$A‘(B‘C)=(A‘B)‘C$	associativity
$A,B\colon C \;\;=\;\; A\colon C \;\wedge\; B\colon C$	antidistributivity
$A\colon B‘C \;\;=\;\; A\colon B \;\wedge\; A\colon C$	distributivity
$A\colon A,B$	generalization
$A‘B\colon A$	specialization
$A\colon A$	reflexivity
$A\colon B \;\wedge\; B\colon A \;\;=\;\; A=B$	antisymmetry
$A\colon B \;\wedge\; B\colon C \;\Rightarrow\; A\colon C$	transitivity
$¢\, null=0$	size

$\text{¢}x = 1$	size
$\text{¢}(A, B) + \text{¢}(A\text{‘}B) = \text{¢}A + \text{¢}B$	size
$\neg\, x: A \;\Rightarrow\; \text{¢}(A\text{‘}x) = 0$	size
$A: B \;\Rightarrow\; \text{¢}A \le \text{¢}B$	size
$A,(A\text{‘}B) \;=\; A$	absorption
$A\text{‘}(A,B) \;=\; A$	absorption
$A: B \;\;=\;\; A,B = B \;\;=\;\; A = A\text{‘}B$	inclusion
$A,(B,C) \;=\; (A,B),(A,C)$	distributivity
$A,(B\text{‘}C) \;=\; (A,B)\text{‘}(A,C)$	distributivity
$A\text{‘}(B,C) \;=\; (A\text{‘}B), (A\text{‘}C)$	distributivity
$A\text{‘}(B\text{‘}C) \;=\; (A\text{‘}B)\text{‘}(A\text{‘}C)$	distributivity
$A: B \;\wedge\; C: D \;\Rightarrow\; A,C: B,D$	conflation, monotonicity
$A: B \;\wedge\; C: D \;\Rightarrow\; A\text{‘}C: B\text{‘}D$	conflation, monotonicity
$null: A$	induction
$A, null = A$	identity
$A \text{ ‘ } null = null$	base
$\text{¢}A = 0 \;\;=\;\; A = null$	size
$x: int \;\wedge\; y: xint \;\wedge\; x{\le}y \;\Rightarrow\; (i: x,..y \;\;=\;\; i: int \;\wedge\; x{\le}i{<}y)$	
$x: int \;\wedge\; y: xint \;\wedge\; x{\le}y \;\Rightarrow\; \text{¢}(x,..y) \;=\; y{-}x$	
$-null \;=\; null$	distribution
$-(A, B) \;=\; -A, -B$	distribution
$A{+}null \;=\; null{+}A \;=\; null$	distribution
$(A, B){+}(C, D) \;=\; A{+}C, A{+}D, B{+}C, B{+}D$	distribution

and similarly for many other operators (see the final page of the book)

End of Bunches

11.4.4 Sets

$\{\sim A\} \;=\; A$

$\sim\{A\} \;=\; A$

$\{A\} \;\ne\; A$

$A \in \{B\} \;\;=\;\; A: B$

$\{A\} \subseteq \{B\} \;\;=\;\; A: B$

$\{A\}: \text{ϟ}B \;\;=\;\; A: B$

$\$\{A\} = \text{¢}A$

$\{A\} \cup \{B\} \;=\; \{A, B\}$

$\{A\} \cap \{B\} \;=\; \{A \text{ ‘ } B\}$

$\{A\} = \{B\} \;\;=\;\; A = B$

$\{A\} \ne \{B\} \;\;=\;\; A \ne B$

End of Sets

11.4.5 Strings

Let S, T, and U be strings; let i and j be items (binary values, numbers, characters, sets, lists, functions); let n be extended natural; let x, y, and z be integers.

$nil; S \;=\; S; nil \;=\; S$

$S; (T; U) \;=\; (S; T); U$

$\leftrightarrow nil \;=\; 0$

$\leftrightarrow i \;=\; 1$

$\leftrightarrow(S; T) \;=\; \leftrightarrow S + \leftrightarrow T$

$S_{nil} \;=\; nil$

$\leftrightarrow S{<}\infty \;\Rightarrow\; (S; i; T)_{\leftrightarrow S} \;=\; i$

$S_{T;\, U} \;=\; S_T; S_U$

$S_{(T_U)} \;=\; (S_T)_U$

$S_{\{A\}} \;=\; \{S_A\}$

$\leftrightarrow S{<}\infty \;\Rightarrow\; nil \;\le\; S \;<\; S; i; T$

$\leftrightarrow S{<}\infty \;\Rightarrow\; (i{<}j \;\Rightarrow\; S; i; T \;<\; S; j; U)$

$\leftrightarrow S{<}\infty \;\Rightarrow\; (i{=}j \;\;=\;\; S; i; T \;=\; S; j; T)$

$0{*}S \;=\; nil$

$(n{+}1){*}S \;=\; n{*}S; S$

$\leftrightarrow S{<}\infty \;\Rightarrow\; S;i;T \triangleleft \leftrightarrow S \triangleright j \;=\; S;j;T$

$x;..x \;=\; nil$

$x;..x{+}1 \;=\; x$

$(x;..y) \;;\; (y;..z) \;=\; x;..z$

$\leftrightarrow(x;..y) \;=\; y{-}x$

End of Strings

11.4.6 Lists

Let S and T be strings; let i and j be items (binary values, numbers, characters, sets, lists, functions); let L, M, and N be lists.

$[S] \neq S$
$\sim[S] = S$
$[\sim L] = L$
$[S]\,T = S_T$
$[S]+[T] = [S;\,T]$
$[S]=[T] \;=\; S=T$
$[S]<[T] \;=\; S<T$
$nil\to i \mid L = i$
$n\to i \mid [S] = [S\triangleleft n\triangleright i]$
$(S;T)\to i \mid L = S\to(T\to i \mid L@S) \mid L$

$\#[S] = \leftrightarrow S$
$S_{[T]} = [S_T]$
$[S]\,[T] = [S_T]$
$L\,\{A\} = \{L\,A\}$
$L\,[S] = [L\,S]$
$(L\,M)\,N = L\,(M\,N)$
$L@nil = L$
$L@i = L\,i$
$L@(S;\,T) = L@S@T$

End of Lists

11.4.7 Functions

Renaming Law — if v and w do not appear in D and w does not appear in b
$\langle v\colon D\to b\rangle = \langle w\colon D\to\langle v\colon D\to b\rangle w\rangle$

Application Law: if element $x\colon D$
$\langle v\colon D\to b\rangle x =$ (substitute x for v in b)

Domain Law
$\square\langle v\colon D\to b\rangle = D$

Law of Extension
$f = \langle v\colon \square f\to f\,v\rangle$

Function Inclusion Law
$f\colon g \;=\; \square g\colon \square f \wedge \forall x\colon \square g\cdot fx\colon gx$

Function Equality Law
$f=g \;=\; \square f=\square g \wedge \forall x\colon \square f\cdot fx=gx$

Laws of Functional Union
$\square(f,g) = \square f \text{‘} \square g$
$(f,g)\,x = f\,x,\,g\,x$

Distributive Laws
$f\,null = null$
$f\,(A,B) = f\,A, f\,B$
$f\,(\S g) = \S y\colon f\,(\square g)\cdot \exists x\colon \square g\cdot fx=y \wedge gx$
f **if** b **then** x **else** y **fi** $=$ **if** b **then** $f\,x$ **else** $f\,y$ **fi**
if b **then** f **else** g **fi** x $=$ **if** b **then** $f\,x$ **else** $g\,x$ **fi**

Function Composition Laws: If $\neg f\colon \square g$
$\square(g\,f) = \S x\colon \square f\cdot fx\colon \square g$
$(g\,f)\,x = g\,(f\,x)$
$f\,(g\,h) = (f\,g)\,h$

Laws of Functional Intersection
$\square(f \text{‘} g) = \square f, \square g$
$(f \text{‘} g)\,x = (f \mid g)\,x \text{‘} (g \mid f)\,x$

Laws of Selective Union
$\square(f \mid g) = \square f, \square g$
$(f \mid g)\,x =$ **if** $x\colon \square f$ **then** $f\,x$ **else** $g\,x$ **fi**
$f \mid f = f$
$f \mid (g \mid h) = (f \mid g) \mid h$
$(g \mid h)\,f = g\,f \mid h\,f$

Arrow Laws
$f\colon null\to A$
$A\to B\colon (A\text{‘}C)\to(B,D)$
$f\colon A\to B \;=\; A\colon \square f \wedge \forall a\colon A\cdot fa\colon B$

End of Functions

11.4.8 Quantifiers

Let x be an element, let a , b and c be binary, let n and m be numeric, let f and g be functions, and let p be a predicate.

$\forall v\colon null\cdot b \;=\; \top$

$\forall v\colon x\cdot b \;=\; \langle v\colon x\to b\rangle\, x$

$\forall v\colon A,B\cdot b \;=\; (\forall v\colon A\cdot b) \wedge (\forall v\colon B\cdot b)$

$\forall v\colon (\S v\colon D\cdot b)\cdot c \;=\; \forall v\colon D\cdot b \Rightarrow c$

$\exists v\colon null\cdot b \;=\; \bot$

$\exists v\colon x\cdot b \;=\; \langle v\colon x\to b\rangle\, x$

$\exists v\colon A,B\cdot b \;=\; (\exists v\colon A\cdot b) \vee (\exists v\colon B\cdot b)$

$\exists v\colon (\S v\colon D\cdot b)\cdot c \;=\; \exists v\colon D\cdot b \wedge c$

$\Sigma v\colon null\cdot n \;=\; 0$

$\Sigma v\colon x\cdot n \;=\; \langle v\colon x\to n\rangle\, x$

$(\Sigma v\colon A,B\cdot n) + (\Sigma v\colon A`B\cdot n) \;=\; (\Sigma v\colon A\cdot n) + (\Sigma v\colon B\cdot n)$

$\Sigma v\colon (\S v\colon D\cdot b)\cdot n \;=\; \Sigma v\colon D\cdot \textbf{if } b \textbf{ then } n \textbf{ else } 0 \textbf{ fi}$

$\Pi v\colon null\cdot n \;=\; 1$

$\Pi v\colon x\cdot n \;=\; \langle v\colon x\to n\rangle\, x$

$(\Pi v\colon A,B\cdot n) \times (\Pi v\colon A`B\cdot n) \;=\; (\Pi v\colon A\cdot n) \times (\Pi v\colon B\cdot n)$

$\Pi v\colon (\S v\colon D\cdot b)\cdot n \;=\; \Pi v\colon D\cdot \textbf{if } b \textbf{ then } n \textbf{ else } 1 \textbf{ fi}$

$MIN\ v\colon null\cdot n \;=\; \infty$

$MIN\ v\colon x\cdot n \;=\; \langle v\colon x\to n\rangle\, x$

$MIN\ v\colon A,B\cdot n \;=\; min\ (MIN\ v\colon A\cdot n)\ (MIN\ v\colon B\cdot n)$

$MIN\ v\colon (\S v\colon D\cdot b)\cdot n \;=\; MIN\ v\colon D\cdot \textbf{if } b \textbf{ then } n \textbf{ else } \infty \textbf{ fi}$

$MAX\ v\colon null\cdot n \;=\; -\infty$

$MAX\ v\colon x\cdot n \;=\; \langle v\colon x\to n\rangle\, x$

$MAX\ v\colon A,B\cdot n \;=\; max\ (MAX\ v\colon A\cdot n)\ (MAX\ v\colon B\cdot n)$

$MAX\ v\colon(\S v\colon D\cdot b)\cdot n \;=\; MAX\ v\colon D\cdot \textbf{if } b \textbf{ then } n \textbf{ else } -\infty \textbf{ fi}$

$\S v\colon null\cdot b \;=\; null$

$\S v\colon x\cdot b \;=\; \textbf{if } \langle v\colon x\to b\rangle\, x \textbf{ then } x \textbf{ else } null \textbf{ fi}$

$\S v\colon A,B\cdot b \;=\; (\S v\colon A\cdot b), (\S v\colon B\cdot b)$

$\S v\colon A`B\cdot b \;=\; (\S v\colon A\cdot b)\ `\ (\S v\colon B\cdot b)$

$\S v\colon (\S v\colon D\cdot b)\cdot c \;=\; \S v\colon D\cdot b \wedge c$

Change of Variable Laws — if d does not appear in b

$\forall r\colon fD\cdot b \;=\; \forall d\colon D\cdot \langle r\colon fD\to b\rangle\ (fd)$

$\exists r\colon fD\cdot b \;=\; \exists d\colon D\cdot \langle r\colon fD\to b\rangle\ (fd)$

$MIN\ r\colon fD\cdot n \;=\; MIN\ d\colon D\cdot \langle r\colon fD\to n\rangle\ (fd)$

$MAX\ r\colon fD\cdot n \;=\; MAX\ d\colon D\cdot \langle r\colon fD\to n\rangle\ (fd)$

Cardinality Law

$¢A \;=\; \Sigma\, (A\to 1)$

Identity Laws

$\forall v\cdot \top$

$\neg\exists v\cdot \bot$

Bunch-Element Conversion Laws

$V\colon W \;=\; \forall v\colon V\cdot \exists w\colon W\cdot v{=}w$

$fV\colon gW \;=\; \forall v\colon V\cdot \exists w\colon W\cdot fv{=}gw$

Idempotent Laws — if $D{\ne}null$ and v does not appear in b

$\forall v\colon D\cdot b \;=\; b$

$\exists v\colon D\cdot b \;=\; b$

Absorption Laws — if $x\colon D$

$\langle v\colon D\to b\rangle\, x \wedge \exists v\colon D\cdot b \;=\; \langle v\colon D\to b\rangle\, x$

$\langle v\colon D\to b\rangle\, x \vee \forall v\colon D\cdot b \;=\; \langle v\colon D\to b\rangle\, x$

$\langle v\colon D\to b\rangle\, x \wedge \forall v\colon D\cdot b \;=\; \forall v\colon D\cdot b$

$\langle v\colon D\to b\rangle\, x \vee \exists v\colon D\cdot b \;=\; \exists v\colon D\cdot b$

Distributive Laws — if $D{\ne}null$ and v does not appear in a

$a \wedge \forall v\colon D\cdot b \;=\; \forall v\colon D\cdot a \wedge b$

$a \wedge \exists v\colon D\cdot b \;=\; \exists v\colon D\cdot a \wedge b$

$a \vee \forall v\colon D\cdot b \;=\; \forall v\colon D\cdot a \vee b$

$a \vee \exists v\colon D\cdot b \;=\; \exists v\colon D\cdot a \vee b$

$a \Rightarrow \forall v\colon D\cdot b \;=\; \forall v\colon D\cdot a \Rightarrow b$

$a \Rightarrow \exists v\colon D\cdot b \;=\; \exists v\colon D\cdot a \Rightarrow b$

Antidistributive Laws — if $D{\ne}null$ and v does not appear in a

$a \Leftarrow \exists v\colon D\cdot b \;=\; \forall v\colon D\cdot a \Leftarrow b$

$a \Leftarrow \forall v\colon D\cdot b \;=\; \exists v\colon D\cdot a \Leftarrow b$

Specialization Law — if $x: D$

$$\forall v: D\cdot\ b \;\;\Longrightarrow\;\; \langle v: D\to b\rangle\ x$$

One-Point Laws — if $x: D$ and v does not appear in x

$$\forall v: D\cdot\ v{=}x \Rightarrow b \;\;=\;\; \langle v: D\to b\rangle\ x$$
$$\exists v: D\cdot\ v{=}x \wedge b \;\;=\;\; \langle v: D\to b\rangle\ x$$

Duality Laws

$$\neg\forall v\cdot\ b \;\;=\;\; \exists v\cdot\ \neg b \quad \text{(deMorgan)}$$
$$\neg\exists v\cdot\ b \;\;=\;\; \forall v\cdot\ \neg b \quad \text{(deMorgan)}$$
$$-\,MAX\ v\cdot\ n \;\;=\;\; MIN\ v\cdot\ {-n}$$
$$-\,MIN\ v\cdot\ n \;\;=\;\; MAX\ v\cdot\ {-n}$$

Solution Laws

$$\S v: D\cdot\ \top \;=\; D$$
$$(\S v: D\cdot\ b):\ D$$
$$\S v: D\cdot\ \bot \;=\; null$$
$$(\S v\cdot\ b): (\S v\cdot\ c) \;\;=\;\; \forall v\cdot\ b\Rightarrow c$$
$$(\S v\cdot\ b), (\S v\cdot\ c) \;\;=\;\; \S v\cdot\ b \vee c$$
$$(\S v\cdot\ b)\ \text{‘}\ (\S v\cdot\ c) \;\;=\;\; \S v\cdot\ b \wedge c$$
$$x: \S p \;\;=\;\; x: \Box p \;\wedge\; px$$
$$\forall f \;=\; (\S f){=}(\Box f)$$
$$\exists f \;=\; (\S f){\ne}null$$

Bounding Laws

if v does not appear in n

$$n > (MAX\ v: D\cdot\ m) \;\Rightarrow\; (\forall v: D\cdot\ n{>}m)$$
$$n < (MIN\ v: D\cdot\ m) \;\Rightarrow\; (\forall v: D\cdot\ n{<}m)$$
$$n \ge (MAX\ v: D\cdot\ m) \;=\; (\forall v: D\cdot\ n{\ge}m)$$
$$n \le (MIN\ v: D\cdot\ m) \;=\; (\forall v: D\cdot\ n{\le}m)$$
$$n \ge (MIN\ v: D\cdot\ m) \;\Leftarrow\; (\exists v: D\cdot\ n{\ge}m)$$
$$n \le (MAX\ v: D\cdot\ m) \;\Leftarrow\; (\exists v: D\cdot\ n{\le}m)$$
$$n > (MIN\ v: D\cdot\ m) \;=\; (\exists v: D\cdot\ n{>}m)$$
$$n < (MAX\ v: D\cdot\ m) \;=\; (\exists v: D\cdot\ n{<}m)$$

Generalization Law — if $x: D$

$$\langle v: D\to b\rangle\ x \;\;\Longrightarrow\;\; \exists v: D\cdot\ b$$

Splitting Laws — for any fixed domain

$$\forall v\cdot\ a \wedge b \;\;=\;\; (\forall v\cdot\ a) \;\wedge\; (\forall v\cdot\ b)$$
$$\exists v\cdot\ a \wedge b \;\;\Longrightarrow\;\; (\exists v\cdot\ a) \;\wedge\; (\exists v\cdot\ b)$$
$$\forall v\cdot\ a \vee b \;\;\Longleftarrow\;\; (\forall v\cdot\ a) \;\vee\; (\forall v\cdot\ b)$$
$$\exists v\cdot\ a \vee b \;\;=\;\; (\exists v\cdot\ a) \;\vee\; (\exists v\cdot\ b)$$
$$\forall v\cdot\ a \Rightarrow b \;\;\Longrightarrow\;\; (\forall v\cdot\ a) \;\Rightarrow\; (\forall v\cdot\ b)$$
$$\forall v\cdot\ a \Rightarrow b \;\;\Longrightarrow\;\; (\exists v\cdot\ a) \;\Rightarrow\; (\exists v\cdot\ b)$$
$$\forall v\cdot\ a = b \;\;\Longrightarrow\;\; (\forall v\cdot\ a) \;=\; (\forall v\cdot\ b)$$
$$\forall v\cdot\ a = b \;\;\Longrightarrow\;\; (\exists v\cdot\ a) \;=\; (\exists v\cdot\ b)$$

Commutative Laws

$$\forall v\cdot\ \forall w\cdot\ b \;\;=\;\; \forall w\cdot\ \forall v\cdot\ b$$
$$\exists v\cdot\ \exists w\cdot\ b \;\;=\;\; \exists w\cdot\ \exists v\cdot\ b$$

Semicommutative Laws (Skolem)

$$\exists v\cdot\ \forall w\cdot\ b \;\;\Longrightarrow\;\; \forall w\cdot\ \exists v\cdot\ b$$
$$\forall x\cdot\ \exists y\cdot\ pxy \;\;=\;\; \exists f\cdot\ \forall x\cdot\ px(fx)$$

Domain Change Laws

$$A: B \;\Longrightarrow\; (\forall v: A\cdot\ b) \;\Leftarrow\; (\forall v: B\cdot\ b)$$
$$A: B \;\Longrightarrow\; (\exists v: A\cdot\ b) \;\Rightarrow\; (\exists v: B\cdot\ b)$$
$$\forall v: A\cdot\ v: B \Rightarrow p \;\;=\;\; \forall v: A\text{‘}B\cdot\ p$$
$$\exists v: A\cdot\ v: B \wedge p \;\;=\;\; \exists v: A\text{‘}B\cdot\ p$$

Extreme Laws

$$(MIN\ n: int\cdot\ n) = (MIN\ n: real\cdot\ n) = -\infty$$
$$(MAX\ n: int\cdot\ n) = (MAX\ n: real\cdot\ n) = \infty$$

Connection Laws (Galois)

$$n{\le}m \;\;=\;\; \forall k\cdot\ \ k{\le}n \Rightarrow k{\le}m$$
$$n{\le}m \;\;=\;\; \forall k\cdot\ \ k{<}n \Rightarrow k{<}m$$
$$n{\le}m \;\;=\;\; \forall k\cdot\ \ m{\le}k \Rightarrow n{\le}k$$
$$n{\le}m \;\;=\;\; \forall k\cdot\ \ m{<}k \Rightarrow n{<}k$$

Distributive Laws — if $D{\ne}null$ and v does not appear in n

$$max\ n\ (MAX\ v: D\cdot\ m) \;=\; (MAX\ v: D\cdot\ max\ n\ m)$$
$$max\ n\ (MIN\ v: D\cdot\ m) \;=\; (MIN\ v: D\cdot\ max\ n\ m)$$
$$min\ n\ (MAX\ v: D\cdot\ m) \;=\; (MAX\ v: D\cdot\ min\ n\ m)$$
$$min\ n\ (MIN\ v: D\cdot\ m) \;=\; (MIN\ v: D\cdot\ min\ n\ m)$$
$$n + (MAX\ v: D\cdot\ m) \;=\; (MAX\ v: D\cdot\ n{+}m)$$
$$n + (MIN\ v: D\cdot\ m) \;=\; (MIN\ v: D\cdot\ n{+}m)$$
$$n - (MAX\ v: D\cdot\ m) \;=\; (MIN\ v: D\cdot\ n{-}m)$$
$$n - (MIN\ v: D\cdot\ m) \;=\; (MAX\ v: D\cdot\ n{-}m)$$
$$(MAX\ v: D\cdot\ m) - n \;=\; (MAX\ v: D\cdot\ m{-}n)$$
$$(MIN\ v: D\cdot\ m) - n \;=\; (MIN\ v: D\cdot\ m{-}n)$$
$$n{\ge}0 \;\Rightarrow\; n \times (MAX\ v: D\cdot\ m) \;=\; (MAX\ v: D\cdot\ n{\times}m)$$
$$n{\ge}0 \;\Rightarrow\; n \times (MIN\ v: D\cdot\ m) \;=\; (MIN\ v: D\cdot\ n{\times}m)$$
$$n{\le}0 \;\Rightarrow\; n \times (MAX\ v: D\cdot\ m) \;=\; (MIN\ v: D\cdot\ n{\times}m)$$
$$n{\le}0 \;\Rightarrow\; n \times (MIN\ v: D\cdot\ m) \;=\; (MAX\ v: D\cdot\ n{\times}m)$$
$$n \times (\Sigma v: D\cdot\ m) \;=\; (\Sigma v: D\cdot\ n{\times}m)$$
$$(\Pi v: D\cdot\ m)^n \;=\; (\Pi v: D\cdot\ m^n)$$

End of Quantifiers

11.4.9 Limits

$(MAX\ m\cdot\ MIN\ n\cdot\ f(m+n))\ \le\ (LIM\ f)\ \le\ (MIN\ m\cdot\ MAX\ n\cdot\ f(m+n))$
$\exists m\cdot\ \forall n\cdot\ p(m+n)\ \Rightarrow\ LIM\ p\ \Rightarrow\ \forall m\cdot\ \exists n\cdot\ p(m+n)$
$(LIM\ n\cdot\ n) = \infty$

End of Limits

11.4.10 Specifications and Programs

For specifications P, Q, R, and S, and binary b,

$ok\ =\ x'=x\ \land\ y'=y\ \land\ ...$
$x{:=}\ e\ =\ x'=e\ \land\ y'=y\ \land\ ...$
$P.\ Q\ =\ \exists x'', y'', ...\cdot\ \langle x', y', ... \rightarrow P\rangle\ x''\ y''\ ...\ \land\ \langle x, y, ... \rightarrow Q\rangle\ x''\ y''\ ...$
$P\|Q\ =\ \exists tP, tQ\cdot\ \langle t'\rightarrow P\rangle tP\ \land\ \langle t'\rightarrow Q\rangle tQ\ \land\ t' = max\ tP\ tQ$
if b **then** P **else** Q **fi** $=\ b \land P \lor \neg b \land Q$
var $x: T\cdot\ P\ =\ \exists x, x': T\cdot\ P$
frame $x\cdot\ P\ =\ P\ \land\ y'=y\ \land\ ...$
while b **do** P **od** $=\ t'\ge t \land$ **if** b **then** $P.\ t{:=}\ t+1.$ **while** b **do** P **od else** ok **fi**
$\forall\sigma, \sigma'\cdot$ **if** b **then** $P.\ W$ **else** ok **fi** $\Leftarrow\ W\ \Rightarrow\ \forall\sigma, \sigma'\cdot$ **while** b **do** P **od** $\Leftarrow\ W$
$\quad (Fmn\ \Leftarrow\ m=n \land ok) \land (Fik\ \Leftarrow\ m\le i<j<k\le n \land (Fij.\ Fjk))$
$\Rightarrow\quad Fmn\ \Leftarrow$ **for** $i{:=}\ m;..n$ **do** $m\le i<n \Rightarrow Fi(i+1)$ **od**
$Im\Rightarrow I'n\ \Leftarrow$ **for** $i{:=}\ m;..n$ **do** $m\le i<n \land Ii \Rightarrow I'(i+1)$ **od**
wait until $w\ =\ t{:=}\ max\ t\ w$
assert b $=$ **if** b **then** ok **else** $print$ “error”. **wait until** ∞ **fi**
ensure $b\ =\ b \land ok$
$P.$ (P **result** e)$=e$ but do not double-prime or substitute in (P **result** e)
$c?\ =\ r{:=}\ r+1$
$c\ =\ \mathbb{M}c_{\,rc-1}$
$c!\ e\ =\ \mathbb{M}c_{\,wc} = e\ \land\ \mathbb{T}c_{\,wc} = t\ \land\ (wc{:=}\ wc+1)$
$\surd c\ =\ \mathbb{T}c_{\,rc}$ + (transit time) $\le t$
ivar $x: T\cdot\ S\ =\ \exists x: time\rightarrow T\cdot\ S$
chan $c: T\cdot\ P\ =\ \exists\mathbb{M}c: \infty{*}T\cdot\ \exists\mathbb{T}c: \infty{*}xreal\cdot$ **var** $rc, wc: xnat := 0\cdot\ P$

$ok.\ P\ =\ P.\ ok\ =\ P$	identity
$P.\ (Q.\ R)\ =\ (P.\ Q).\ R$	associativity
$P\lor Q.\ R\lor S\ =\ (P.\ R) \lor (P.\ S) \lor (Q.\ R) \lor (Q.\ S)$	distributivity
if b **then** P **else** Q **fi**$.\ R$ $=$ **if** b **then** $P.\ R$ **else** $Q.\ R$ **fi**	distributivity (unprimed b)
$P.$ **if** b **then** Q **else** R **fi** $=$ **if** $P.\ b$ **then** $P.\ Q$ **else** $P.\ R$ **fi**	distributivity (unprimed b)
$P \| Q\ =\ Q \| P$	symmetry
$P \| (Q \| R)\ =\ (P \| Q) \| R$	associativity
$P \| t'=t\ =\ P\ =\ t'=t \| P$	identity
$P \| Q\lor R\ =\ (P \| Q) \lor (P \| R)$	distributivity
$P \|$ **if** b **then** Q **else** R **fi** $=$ **if** b **then** $P \| Q$ **else** $P \| R$ **fi**	distributivity
if b **then** $P\|Q$ **else** $R\|S$ **fi** $=$ **if** b **then** P **else** R **fi** $\|$ **if** b **then** Q **else** S **fi**	distributivity
$x{:=}$ **if** b **then** e **else** f **fi** $=$ **if** b **then** $x{:=}\ e$ **else** $x{:=}\ f$ **fi**	functional-imperative

End of Specifications and Programs

11.4.11 Substitution

Let x and y be different boundary state variables, let e and f be expressions of the prestate, and let P be a specification.

$x{:=}\,e.\,P \;=\;$ (for x substitute e in P)

$(x{:=}\,e \,\|\, y{:=}f).\; P \;=\;$ (for x substitute e and independently for y substitute f in P)

End of Substitution

11.4.12 Conditions

Let P and Q be any specifications, and let C be a precondition, and let C' be the corresponding postcondition (in other words, C' is the same as C but with primes on all the state variables).

$C \wedge (P.\,Q) \;\Leftarrow\; C{\wedge}P.\,Q$

$C \Rightarrow (P.Q) \;\Leftarrow\; C{\Rightarrow}P.\,Q$

$(P.Q) \wedge C' \;\Leftarrow\; P.\,Q{\wedge}C'$

$(P.Q) \Leftarrow C' \;\Leftarrow\; P.\,Q{\Leftarrow}C'$

$P.\,C{\wedge}Q \;\Leftarrow\; P{\wedge}C'.\,Q$

$P.\,Q \;\Leftarrow\; P{\wedge}C'.\; C{\Rightarrow}Q$

C is a sufficient precondition for P to be refined by S
if and only if $C{\Rightarrow}P$ is refined by S .

C' is a sufficient postcondition for P to be refined by S
if and only if $C'{\Rightarrow}P$ is refined by S .

End of Conditions

11.4.13 Refinement

Refinement by Steps (Stepwise Refinement) (monotonicity, transitivity)

If $A \Leftarrow$ **if** b **then** C **else** D **fi** and $C \Leftarrow E$ and $D \Leftarrow F$ are theorems,
then $A \Leftarrow$ **if** b **then** E **else** F **fi** is a theorem.

If $A \Leftarrow B.C$ and $B \Leftarrow D$ and $C \Leftarrow E$ are theorems, then $A \Leftarrow D.E$ is a theorem.

If $A \Leftarrow B\|C$ and $B \Leftarrow D$ and $C \Leftarrow E$ are theorems, then $A \Leftarrow D\|E$ is a theorem.

If $A \Leftarrow B$ and $B \Leftarrow C$ are theorems, then $A \Leftarrow C$ is a theorem.

Refinement by Parts (monotonicity, conflation)

If $A \Leftarrow$ **if** b **then** C **else** D **fi** and $E \Leftarrow$ **if** b **then** F **else** G **fi** are theorems,
then $A{\wedge}E \Leftarrow$ **if** b **then** $C{\wedge}F$ **else** $D{\wedge}G$ **fi** is a theorem.

If $A \Leftarrow B.C$ and $D \Leftarrow E.F$ are theorems, then $A{\wedge}D \Leftarrow B{\wedge}E.\,C{\wedge}F$ is a theorem.

If $A \Leftarrow B\|C$ and $D \Leftarrow E\|F$ are theorems, then $A{\wedge}D \Leftarrow B{\wedge}E \,\|\, C{\wedge}F$ is a theorem.

If $A \Leftarrow B$ and $C \Leftarrow D$ are theorems, then $A{\wedge}C \Leftarrow B{\wedge}D$ is a theorem.

Refinement by Cases

$P \Leftarrow$ **if** b **then** Q **else** R **fi** is a theorem if and only if
$P \Leftarrow b \wedge Q$ and $P \Leftarrow \neg b \wedge R$ are theorems.

End of Refinement

End of Laws

11.5 Names

$\mathit{abs}\colon\ \mathit{xreal} \to \S r\colon \mathit{xreal} \cdot r \ge 0$	$\mathit{abs}\ r \ = \ \textbf{if}\ r \ge 0\ \textbf{then}\ r\ \textbf{else}\ -r\ \textbf{fi}$
bin (the binary values)	$\mathit{bin} \ = \ \top, \bot$
$\mathit{ceil}\colon\ \mathit{real} \to \mathit{int}$	$r \le \mathit{ceil}\ r < r+1$
char (the characters)	$\mathit{char} \ = \ \ldots, \text{“a”}, \text{“A”}, \ldots$
$\mathit{div}\colon\ \mathit{real} \to (\S r\colon \mathit{real} \cdot r > 0) \to \mathit{int}$	$\mathit{div}\ x\ y \ = \ \mathit{floor}\ (x/y)$
$\mathit{divides}\colon\ (\mathit{nat}+1) \to \mathit{int} \to \mathit{bin}$	$\mathit{divides}\ n\ i \ \ = \ \ i/n\colon \mathit{int}$
$\mathit{entro}\colon\ \mathit{prob} \to \S r\colon \mathit{xreal} \cdot r \ge 0$	$\mathit{entro}\ p \ = \ p \times \mathit{info}\ p \ + \ (1-p) \times \mathit{info}\ (1-p)$
$\mathit{even}\colon\ \mathit{int} \to \mathit{bin}$	$\mathit{even}\ i \ \ = \ \ i/2\colon \mathit{int}$
	$\mathit{even} \ = \ \mathit{divides}\ 2$
$\mathit{floor}\colon\ \mathit{real} \to \mathit{int}$	$\mathit{floor}\ r \le r < \mathit{floor}\ r + 1$
$\mathit{info}\colon\ \mathit{prob} \to \S r\colon \mathit{xreal} \cdot r \ge 0$	$\mathit{info}\ p \ = \ -\log p$
int (the integers)	$\mathit{int} \ = \ \mathit{nat}, -\mathit{nat}$
LIM (limit quantifier)	see Laws
$\mathit{log}\colon\ (\S r\colon \mathit{xreal} \cdot r \ge 0) \to \mathit{xreal}$	$\log\ (2^x) = x$
	$\log\ (x \times y) \ = \ \log x + \log y$
$\mathit{max}\colon\ \mathit{xrat} \to \mathit{xrat} \to \mathit{xrat}$	$\mathit{max}\ x\ y \ = \ \textbf{if}\ x \ge y\ \textbf{then}\ x\ \textbf{else}\ y\ \textbf{fi}$
	$-\mathit{max}\ a\ b \ = \ \mathit{min}\ (-a)\ (-b)$
MAX (maximum quantifier)	see Laws
$\mathit{min}\colon\ \mathit{xrat} \to \mathit{xrat} \to \mathit{xrat}$	$\mathit{min}\ x\ y \ = \ \textbf{if}\ x \le y\ \textbf{then}\ x\ \textbf{else}\ y\ \textbf{fi}$
	$-\mathit{min}\ a\ b \ = \ \mathit{max}\ (-a)\ (-b)$
MIN (minimum quantifier)	see Laws
$\mathit{mod}\colon\ \mathit{real} \to (\S r\colon \mathit{real} \cdot r > 0) \to \mathit{real}$	$0 \le \mathit{mod}\ a\ d < d$
	$a \ = \ \mathit{div}\ a\ d \times d \ + \ \mathit{mod}\ a\ d$
nat (the naturals)	$0, \mathit{nat}+1\colon \mathit{nat}$
	$0, B+1\colon B \ \Rightarrow \ \mathit{nat}\colon B$
nil (the empty string)	$\leftrightarrow \mathit{nil} \ = \ 0$
	$\mathit{nil}; S \ = \ S \ = \ S; \mathit{nil}$
	$\mathit{nil} \le S$
null (the empty bunch)	$¢\mathit{null} \ = \ 0$
	$\mathit{null}, A \ = \ A \ = \ A, \mathit{null}$
	$\mathit{null}\colon A$
$\mathit{odd}\colon\ \mathit{int} \to \mathit{bin}$	$\mathit{odd}\ i \ \ = \ \ \neg\ i/2\colon \mathit{int}$
	$\mathit{odd} \ = \ \neg \mathit{even}$
ok (the empty program)	$\mathit{ok} \ \ = \ \ \sigma' = \sigma$
	$\mathit{ok}.P \ \ = \ \ P \ \ = \ \ P.\mathit{ok}$
prob (probability)	$\mathit{prob} \ = \ \S r\colon \mathit{real} \cdot 0 \le r \le 1$
rand (random number)	$\mathit{rand}\ n\colon 0,..n$
rat (the rationals)	$\mathit{rat} \ = \ \mathit{int}/(\mathit{nat}+1)$
real (the reals)	$r\colon \mathit{real} \ \ = \ \ r\colon \mathit{xreal} \ \wedge \ -\infty < r < \infty$
$\mathit{suc}\colon\ \mathit{nat} \to (\mathit{nat}+1)$	$\mathit{suc}\ n = n+1$
xint (the extended integers)	$\mathit{xint} \ = \ -\infty, \mathit{int}, \infty$
xnat (the extended naturals)	$\mathit{xnat} \ = \ \mathit{nat}, \infty$
xrat (the extended rationals)	$\mathit{xrat} \ = \ -\infty, \mathit{rat}, \infty$
xreal (the extended reals)	$x\colon \mathit{xreal} \ \ = \ \ \exists f\colon \mathit{nat} \to \mathit{rat} \cdot x = \mathit{LIM}\ f$

End of Names

11.6 Symbols

11.7 Precedence

0	⊤ ⊥ ( ) { } [ ] ⟨ ⟩ **if fi** **do od** number text name superscript subscript
1	@ juxtaposition
2	prefix– ¢ $ ↔ # * ~ ↯ □ → √
3	× / ∩
4	+ infix– ⁺ ∪
5	; ;.. ‘
6	, ,.. | ⊲ ⊳
7	= ⧧ < > ≤ ≥ : :: ∈ ⊆
8	¬
9	∧
10	∨
11	⇒ ⇐
12	:= ! ?
13	**exit when** **go to** **wait until** **assert** **ensure** **or**
14	. ‖ **result**
15	∀· ∃· Σ· Π· §· *LIM*· *MAX*· *MIN*· **var**· **ivar**· **chan**· **frame**·
16	**=** **⟹** **⟸**

Superscripting and subscripting serve to bracket all operations within them.

Juxtaposition associates from left to right, so $a\ b\ c$ means the same as $(a\ b)\ c$. The infix operators @ / – associate from left to right. The infix operators * → associate from right to left. The infix operators × ∩ + ⁺ ∪ ; ‘ , | ∧ ∨ . ‖ are associative (they associate in both directions).

On levels 7, 11, and 16 the operators are continuing. For example, $a=b=c$ neither associates to the left nor associates to the right, but means the same as $a=b \wedge b=c$. On any one of these levels, a mixture of continuing operators can be used. For example, $a \le b < c$ means the same as $a \le b \wedge b < c$.

The operators **=** **⟹** **⟸** are identical to = ⇒ ⇐ except for precedence.

End of Precedence

11.8 Distribution

The operators in the following expressions distribute over bunch union in any operand:

$[A]$ $A@B$ $A\ B$ $-A$ $\$A$ $\leftrightarrow A$ $\#A$ $\sim A$

A^B A_B $A \times B$ A/B $A \cap B$ $A+B$ $A-B$ $A^{+}B$ $A \cup B$ $A;B$ $A‘B$

$\neg A$ $A \wedge B$ $A \vee B$

The operator in $A*B$ distributes over bunch union in its left operand only.

End of Distribution

End of Reference

End of a Practical Theory of Programming