a Practical Theory of Programming

2018-5-17 edition

Eric C.R. Hehner

Department of Computer Science
University of Toronto
Toronto ON M5S 2E4 Canada

The first edition of this book was published by
Springer-Verlag Publishers, New York, 1993
ISBN 0-387-94106-1   QA76.6.H428

The current edition is available free at
www.cs.utoronto.ca/~hehner/aPToP

An on-line course based on this book is at
www.cs.utoronto.ca/~hehner/FMSD

The author's website is
www.cs.utoronto.ca/~hehner

11.4 Laws

11.4.0 Binary

Let a , b , c , d , and e be binary.

Binary Laws

$\top$

$\neg\bot$

Law of Excluded Middle (Tertium non Datur)

$a \lor \neg a$

Law of Noncontradiction

$\neg(a \land \neg a)$

Base Laws

$\neg(a \land \bot)$

$a \lor \top$

$a \Rightarrow \top$

$\bot \Rightarrow a$

Identity Laws

$\top \land a \;\;=\;\; a$

$\bot \lor a \;\;=\;\; a$

$\top \Rightarrow a \;\;=\;\; a$

$\top = a \;\;=\;\; a$

Idempotent Laws

$a \land a \;\;=\;\; a$

$a \lor a \;\;=\;\; a$

Reflexive Laws

$a \Rightarrow a$

$a = a$

Laws of Indirect Proof

$\neg a \Rightarrow \bot \;\;=\;\; a$ (Reductio ad Absurdum)

$\neg a \Rightarrow a \;\;=\;\; a$

Law of Specialization

$a \land b \Rightarrow a$

Associative Laws

$a \land (b \land c) \;\;=\;\; (a \land b) \land c$

$a \lor (b \lor c) \;\;=\;\; (a \lor b) \lor c$

$a = (b = c) \;\;=\;\; (a = b) = c$

$a \ne (b \ne c) \;\;=\;\; (a \ne b) \ne c$

$a = (b \ne c) \;\;=\;\; (a = b) \ne c$

Mirror Law

$a \Leftarrow b \;\;=\;\; b \Rightarrow a$

Law of Double Negation

$\neg\neg a \;\;=\;\; a$

Duality Laws (deMorgan)

$\neg(a \land b) \;\;=\;\; \neg a \lor \neg b$

$\neg(a \lor b) \;\;=\;\; \neg a \land \neg b$

Laws of Exclusion

$a \Rightarrow \neg b \;\;=\;\; b \Rightarrow \neg a$

$a = \neg b \;\;=\;\; a \ne b \;\;=\;\; \neg a = b$

Laws of Inclusion

$a \Rightarrow b \;\;=\;\; \neg a \lor b$ (Material Implication)

$a \Rightarrow b \;\;=\;\; (a \land b \;\;=\;\; a)$

$a \Rightarrow b \;\;=\;\; (a \lor b \;\;=\;\; b)$

Absorption Laws

$a \land (a \lor b) \;\;=\;\; a$

$a \lor (a \land b) \;\;=\;\; a$

Laws of Direct Proof

$(a \Rightarrow b) \land a \Rightarrow b$ (Modus Ponens)

$(a \Rightarrow b) \land \neg b \Rightarrow \neg a$ (Modus Tollens)

$(a \lor b) \land \neg a \Rightarrow b$(Disjunctive Syllogism)

Transitive Laws

$(a \land b) \land (b \land c) \Rightarrow (a \land c)$

$(a \Rightarrow b) \land (b \Rightarrow c) \Rightarrow (a \Rightarrow c)$

$(a = b) \land (b = c) \Rightarrow (a = c)$

$(a \Rightarrow b) \land (b = c) \Rightarrow (a \Rightarrow c)$

$(a = b) \land (b \Rightarrow c) \Rightarrow (a \Rightarrow c)$

Distributive Laws (Factoring)

$a \land (b \land c) \;\;=\;\; (a \land b) \land (a \land c)$

$a \land (b \lor c) \;\;=\;\; (a \land b) \lor (a \land c)$

$a \lor (b \land c) \;\;=\;\; (a \lor b) \land (a \lor c)$

$a \lor (b \lor c) \;\;=\;\; (a \lor b) \lor (a \lor c)$

$a \lor (b \Rightarrow c) \;\;=\;\; (a \lor b) \Rightarrow (a \lor c)$

$a \lor (b = c) \;\;=\;\; (a \lor b) = (a \lor c)$

$a \Rightarrow (b \land c) \;\;=\;\; (a \Rightarrow b) \land (a \Rightarrow c)$

$a \Rightarrow (b \lor c) \;\;=\;\; (a \Rightarrow b) \lor (a \Rightarrow c)$

$a \Rightarrow (b \Rightarrow c) \;\;=\;\; (a \Rightarrow b) \Rightarrow (a \Rightarrow c)$

$a \Rightarrow (b = c) \;\;=\;\; (a \Rightarrow b) = (a \Rightarrow c)$

Symmetry Laws (Commutative Laws)

$$a \land b \;=\; b \land a$$
$$a \lor b \;=\; b \lor a$$
$$a = b \;=\; b = a$$
$$a \ne b \;=\; b \ne a$$

Antisymmetry Law (Double Implication)

$$(a \Rightarrow b) \land (b \Rightarrow a) \;=\; a = b$$

Laws of Discharge

$$a \land (a \Rightarrow b) \;=\; a \land b$$
$$a \Rightarrow (a \land b) \;=\; a \Rightarrow b$$

Antimonotonic Law

$$a \Rightarrow b \;\Longrightarrow\; (b \Rightarrow c) \Rightarrow (a \Rightarrow c)$$

Monotonic Laws

$$a \Rightarrow b \;\Longrightarrow\; c \land a \Rightarrow c \land b$$
$$a \Rightarrow b \;\Longrightarrow\; c \lor a \Rightarrow c \lor b$$
$$a \Rightarrow b \;\Longrightarrow\; (c \Rightarrow a) \Rightarrow (c \Rightarrow b)$$

Law of Generalization

$$a \Rightarrow a \lor b$$

Antidistributive Laws

$$a \land b \Rightarrow c \;=\; (a \Rightarrow c) \lor (b \Rightarrow c)$$
$$a \lor b \Rightarrow c \;=\; (a \Rightarrow c) \land (b \Rightarrow c)$$

Laws of Portation

$$a \land b \Rightarrow c \;=\; a \Rightarrow (b \Rightarrow c)$$
$$a \land b \Rightarrow c \;=\; a \Rightarrow \neg b \lor c$$

Laws of Conflation

$$(a \Rightarrow b) \land (c \Rightarrow d) \;\Longrightarrow\; a \land c \Rightarrow b \land d$$
$$(a \Rightarrow b) \land (c \Rightarrow d) \;\Longrightarrow\; a \lor c \Rightarrow b \lor d$$

Contrapositive Law

$$a \Rightarrow b \;=\; \neg b \Rightarrow \neg a$$

Laws of Equality and Difference

$$a{=}b \;=\; (a \land b) \lor (\neg a \land \neg b)$$
$$a{\ne}b \;=\; (a \land \neg b) \lor (\neg a \land b)$$

Law of Resolution

$$a \land c \;\Longrightarrow\; (a \lor b) \land (\neg b \lor c) \;=\; (a \land \neg b) \lor (b \land c) \;\Longrightarrow\; a \lor c$$

Case Creation Laws

a = **if** b **then** $b{\Rightarrow}a$ **else** $\neg b \Rightarrow a$ **fi**
a = **if** b **then** $b{\land}a$ **else** $\neg b \land a$ **fi**
a = **if** b **then** $b{=}a$ **else** $b{\ne}a$ **fi**

Case Analysis Laws

if a **then** b **else** c **fi** $=$ $(a \land b) \lor (\neg a \land c)$
if a **then** b **else** c **fi** $=$ $(a{\Rightarrow}b) \land (\neg a \Rightarrow c)$

One Case Laws

if a **then** $\top$ **else** b **fi** $=$ $a{\lor}b$
if a **then** $\bot$ **else** b **fi** $=$ $\neg a \land b$
if a **then** b **else** $\top$ **fi** $=$ $a{\Rightarrow}b$
if a **then** b **else** $\bot$ **fi** $=$ $a{\land}b$

Case Absorption Laws

if a **then** b **else** c **fi** = **if** a **then** $a{\land}b$ **else** c **fi**
if a **then** b **else** c **fi** = **if** a **then** $a{\Rightarrow}b$ **else** c **fi**
if a **then** b **else** c **fi** = **if** a **then** $a{=}b$ **else** c **fi**
if a **then** b **else** c **fi** = **if** a **then** b **else** $\neg a \land c$ **fi**
if a **then** b **else** c **fi** = **if** a **then** b **else** $a{\lor}c$ **fi**
if a **then** b **else** c **fi** = **if** a **then** b **else** $a{\ne}c$ **fi**

Case Distributive Laws (Case Factoring)

$\neg$ **if** a **then** b **else** c **fi** $=$ **if** a **then** $\neg b$ **else** $\neg c$ **fi**
if a **then** b **else** c **fi** $\land\, d$ $=$ **if** a **then** $b{\land}d$ **else** $c{\land}d$ **fi**
and similarly replacing $\land$ by any of $\lor$ $=$ $\ne$ $\Rightarrow$ $\Leftarrow$
if a **then** $b \land c$ **else** $d \land e$ **fi** $=$ **if** a **then** b **else** d **fi** $\land$ **if** a **then** c **else** e **fi**
and similarly replacing $\land$ by any of $\lor$ $=$ $\ne$ $\Rightarrow$ $\Leftarrow$

End of Binary

11.4.1 Generic

The operators $=\ \neq$ **if then else fi** apply to every type of expression (but the first operand of **if then else fi** must be binary), with the laws

$x = x$	reflexivity	**if** $\top$ **then** x **else** y **fi** $= x$	case base
$x=y \quad = \quad y=x$	symmetry	**if** $\bot$ **then** x **else** y **fi** $= y$	case base
$x=y \land y=z \;\Rightarrow\; x=z$	transitivity	**if** a **then** x **else** x **fi** $= x$	case idempotent
$x=y \;\Rightarrow\; f\,x = f\,y$	transparency	**if** a **then** x **else** y **fi** $=$ **if** $\neg a$ **then** y **else** x **fi**	case reversal
$x \neq y \quad = \quad \neg(x=y)$	unequality		

The operators $< \le > \ge$ apply to numbers, characters, strings, and lists, with the laws

$\neg\, x<x$	irreflexivity
$\neg(x<y \land x>y)$	exclusivity
$\neg(x<y \land x=y)$	exclusivity
$x\le y \land y\le x \;=\; x=y$	antisymmetry
$x<y \land y<z \Rightarrow x<z$	transitivity
$x\le y \quad = \quad x<y \lor x=y$	inclusivity
$x>y \quad = \quad y<x$	mirror
$x\ge y \quad = \quad y\le x$	mirror
$x<y \lor x=y \lor x>y$	totality, trichotomy

End of Generic

11.4.2 Numbers

Let d be a sequence of (zero or more) digits, and let x , y , and z be numbers.

$d0+1 = d1$	counting
$d1+1 = d2$	counting
$d2+1 = d3$	counting
$d3+1 = d4$	counting
$d4+1 = d5$	counting
$d5+1 = d6$	counting
$d6+1 = d7$	counting
$d7+1 = d8$	counting
$d8+1 = d9$	counting
$d9+1 = (d+1)0$	counting (see Exercise 32)
$x+0 = x$	identity
$x+y = y+x$	symmetry
$x+(y+z) \;=\; (x+y)+z$	associativity
$-\infty<x<\infty \Rightarrow (x+y = x+z \quad = \quad y=z)$	cancellation
$-\infty<x \Rightarrow \;\infty+x = \infty$	absorption
$x<\infty \;\Rightarrow\; -\infty + x = -\infty$	absorption
$-x \;=\; 0 - x$	negation
$- -x = x$	self-inverse
$-(x+y) = -x + -y$	distributivity
$-(x-y) = x-y$	antisymmetry
$-(x\times y) = -x \times y$	semi-distributivity
$-(x/y) = -x / y$	semi-distributivity
$x-0 = x$	identity
$x-y \;=\; x + -y$	subtraction
$x + (y - z) \;=\; (x + y) - z$	associativity

$-\infty<x<\infty \Rightarrow (x-y = x-z \;=\; y=z)$	cancellation
$-\infty<x<\infty \Rightarrow x-x = 0$	inverse
$x<\infty \Rightarrow \infty-x = \infty$	absorption
$-\infty<x \Rightarrow -\infty - x = -\infty$	absorption
$-\infty<x<\infty \Rightarrow x\times 0 = 0$	base
$x\times 1 = x$	identity
$x\times y = y\times x$	symmetry
$x\times(y+z) = x\times y + x\times z$	distributivity
$x\times(y\times z) = (x\times y)\times z$	associativity
$-\infty<x<\infty \wedge x\ne 0 \Rightarrow (x\times y = x\times z \;=\; y=z)$	cancellation
$0<x \Rightarrow x\times\infty = \infty$	absorption
$0<x \Rightarrow x \times -\infty = -\infty$	absorption
$x/1 = x$	identity
$-\infty<x<\infty \wedge x\ne 0 \Rightarrow x/x = 1$	inverse
$x\times(y/z) = (x\times y)/z = x/(z/y)$	multiplication-division
$y\ne 0 \Rightarrow (x/y)/z = x/(y\times z)$	multiplication-division
$-\infty<x<\infty \Rightarrow x/\infty = 0 = x/-\infty$	annihilation
$-\infty<x<\infty \Rightarrow x^0 = 1$	base
$x^1 = x$	identity
$x^{y+z} = x^y \times x^z$	exponents
$x^{y\times z} = (x^y)^z$	exponents
$-\infty<0<1<\infty$	direction
$x<y \;=\; -y<-x$	reflection
$-\infty<x<\infty \Rightarrow (x+y < x+z \;=\; y<z)$	cancellation, translation
$0<x<\infty \Rightarrow (x\times y < x\times z \;=\; y<z)$	cancellation, scale
$x<y \vee x=y \vee x>y$	trichotomy
$-\infty \le x \le \infty$	extremes

End of Numbers

11.4.3 Bunches

Let x and y be elements (binary values, numbers, characters, sets, strings and lists of elements).

$x\text{: } y \;=\; x=y$	elementary law
$x\text{: } A,B \;=\; x\text{: } A \vee x\text{: } B$	compound law
$A,A = A$	idempotence
$A,B = B,A$	symmetry
$A,(B,C) = (A,B),C$	associativity
$A\text{‘}A = A$	idempotence
$A\text{‘}B = B\text{‘}A$	symmetry
$A\text{‘}(B\text{‘}C) = (A\text{‘}B)\text{‘}C$	associativity
$A,B\text{: } C \;=\; A\text{: } C \wedge B\text{: } C$	antidistributivity
$A\text{: } B\text{‘}C \;=\; A\text{: } B \wedge A\text{: } C$	distributivity
$A\text{: } A,B$	generalization
$A\text{‘}B\text{: } A$	specialization
$A\text{: } A$	reflexivity
$A\text{: } B \wedge B\text{: } A \;=\; A=B$	antisymmetry
$A\text{: } B \wedge B\text{: } C \Rightarrow A\text{: } C$	transitivity
$\text{¢}\, null = 0$	size
$\text{¢}x = 1$	size
$\text{¢}(A, B) + \text{¢}(A\text{‘}B) = \text{¢}A + \text{¢}B$	size

$\neg\ x{:}\ A \;\Rightarrow\; ¢(A‘x) = 0$	size
$A{:}\ B \;\Rightarrow\; ¢A \le ¢B$	size
$A,(A‘B) \;=\; A$	absorption
$A‘(A,B) \;=\; A$	absorption
$A{:}\ B \;\boldsymbol{=}\; A,B = B \;\boldsymbol{=}\; A = A‘B$	inclusion
$A,(B,C) \;=\; (A,B),(A,C)$	distributivity
$A,(B‘C) \;=\; (A,B)‘(A,C)$	distributivity
$A‘(B,C) \;=\; (A‘B), (A‘C)$	distributivity
$A‘(B‘C) \;=\; (A‘B)‘(A‘C)$	distributivity
$A{:}\ B \;\wedge\; C{:}\ D \;\Rightarrow\; A,C{:}\ B,D$	conflation, monotonicity
$A{:}\ B \;\wedge\; C{:}\ D \;\Rightarrow\; A‘C{:}\ B‘D$	conflation, monotonicity
$null{:}\ A$	induction
$A, null = A$	identity
$A ‘ null = null$	base
$¢A = 0 \;\boldsymbol{=}\; A = null$	size
$x{:}\ int \;\wedge\; y{:}\ xint \;\wedge\; x{\le}y \;\Rightarrow\; (i{:}\ x,..y \;\boldsymbol{=}\; i{:}\ int \;\wedge\; x{\le}i{<}y)$	
$x{:}\ int \;\wedge\; y{:}\ xint \;\wedge\; x{\le}y \;\Rightarrow\; ¢(x,..y) \;=\; y{-}x$	
$-null \;=\; null$	distribution
$-(A, B) \;=\; -A, -B$	distribution
$A{+}null \;=\; null{+}A \;=\; null$	distribution
$(A, B){+}(C, D) \;=\; A{+}C, A{+}D, B{+}C, B{+}D$	distribution

and similarly for many other operators (see the final page of the book)

End of Bunches

11.4.4 Sets

$\{{\sim}A\} \;=\; A$

${\sim}\{A\} \;=\; A$

$\{A\} \;\neq\; A$

$A \in \{B\} \;\boldsymbol{=}\; A{:}\ B$

$\{A\} \subseteq \{B\} \;\boldsymbol{=}\; A{:}\ B$

$\{A\}{:}\ \lightning B \;\boldsymbol{=}\; A{:}\ B$

$\$\{A\} = ¢A$

$\{A\} \cup \{B\} \;=\; \{A, B\}$

$\{A\} \cap \{B\} \;=\; \{A ‘ B\}$

$\{A\} = \{B\} \;\boldsymbol{=}\; A = B$

$\{A\} \neq \{B\} \;\boldsymbol{=}\; A \neq B$

End of Sets

11.4.5 Strings

Let S, T, and U be strings; let i and j be items (binary values, numbers, characters, sets, lists, functions); let n be extended natural; let x, y, and z be integers.

$nil;\ S \;=\; S;\ nil \;=\; S$

$S;\ (T;\ U) \;=\; (S;\ T);\ U$

$\leftrightarrow nil \;=\; 0$

$\leftrightarrow i \;=\; 1$

$\leftrightarrow (S;\ T) \;=\; \leftrightarrow S + \leftrightarrow T$

$S_{nil} \;=\; nil$

$\leftrightarrow S{<}\infty \;\Rightarrow\; (S;\ i;\ T)_{\leftrightarrow S} \;=\; i$

$S_{T;\ U} \;=\; S_T;\ S_U$

$S_{(T_U)} \;=\; (S_T)_U$

$S_{\{A\}} \;=\; \{S_A\}$

$\leftrightarrow S{<}\infty \;\Rightarrow\; nil \;\le\; S \;<\; S;\ i;\ T$

$\leftrightarrow S{<}\infty \;\Rightarrow\; (i{<}j \;\Rightarrow\; S;\ i;\ T \;<\; S;\ j;\ U)$

$\leftrightarrow S{<}\infty \;\Rightarrow\; (i{=}j \;\boldsymbol{=}\; S;\ i;\ T \;=\; S;\ j;\ T)$

$0{*}S \;=\; nil$

$(n{+}1){*}S \;=\; n{*}S;\ S$

$\leftrightarrow S{<}\infty \;\Rightarrow\; S;i;T \triangleleft \leftrightarrow S \triangleright j \;=\; S;j;T$

$x;..x \;=\; nil$

$x;..x{+}1 \;=\; x$

$(x;..y)\ ;\ (y;..z) \;=\; x;..z$

$\leftrightarrow (x;..y) \;=\; y{-}x$

End of Strings

11.4.6 Lists

Let S and T be strings; let i and j be items (binary values, numbers, characters, sets, lists, functions); let L, M, and N be lists.

$[S] \ne S$

$\sim[S] = S$

$[\sim L] = L$

$[S]\ T = S_T$

$[S]+[T] = [S;\ T]$

$[S] = [T] \;=\; S = T$

$[S] < [T] \;=\; S < T$

$nil \rightarrow i \mid L = i$

$n \rightarrow i \mid [S] = [S \triangleleft n \triangleright i]$

$(S;T) \rightarrow i \mid L = S \rightarrow (T \rightarrow i \mid L@S) \mid L$

$\#[S] = \leftrightarrow S$

$S_{[T]} = [S_T]$

$[S]\ [T] = [S_T]$

$L\ \{A\} = \{L\ A\}$

$L\ [S] = [L\ S]$

$(L\ M)\ N = L\ (M\ N)$

$L@nil = L$

$L@i = L\ i$

$L@(S;\ T) = L@S@T$

End of Lists

11.4.7 Functions

Renaming Law — if v and w do not appear in D and w does not appear in b

$\langle v: D \rightarrow b\rangle = \langle w: D \rightarrow \langle v: D \rightarrow b\rangle w\rangle$

Application Law: if element $x: D$

$\langle v: D \rightarrow b\rangle x =$ (substitute x for v in b)

Law of Extension

$f = \langle v: \Box f \rightarrow f\ v\rangle$

Domain Law

$\Box\langle v: D \rightarrow b\rangle = D$

Function Inclusion Law

$f: g \;=\; \Box g: \Box f \land \forall x: \Box g \cdot fx: gx$

Function Equality Law

$f = g \;=\; \Box f = \Box g \land \forall x: \Box f \cdot fx = gx$

Laws of Functional Union

$\Box(f, g) = \Box f \text{ ‘ } \Box g$

$(f, g)\ x = f\,x, g\,x$

Laws of Selective Union

$f \mid f = f$

$(g \mid h)\ f = g\,f \mid h\,f$

$\langle v: A \rightarrow x\rangle \mid \langle v: B \rightarrow y\rangle = \langle v: A, B \rightarrow \textbf{if } v: A \textbf{ then } x \textbf{ else } y \textbf{ fi}\rangle$

Function Composition Laws: If $\neg f: \Box g$

$\Box(g\,f) = \S x: \Box f \cdot fx: \Box g$

$(g\,f)\ x = g\ (f\,x)$

$f\ (g\,h) = (f\,g)\ h$

Cardinality Law

$¢A = \Sigma\ (A \rightarrow 1)$

Laws of Functional Intersection

$\Box(f \text{ ‘ } g) = \Box f, \Box g$

$(f \text{ ‘ } g)\ x = (f \mid g)\ x \text{ ‘ } (g \mid f)\ x$

Laws of Selective Union

$\Box(f \mid g) = \Box f, \Box g$

$(f \mid g)\ x = \textbf{if } x: \Box f \textbf{ then } f\,x \textbf{ else } g\,x \textbf{ fi}$

$f \mid (g \mid h) = (f \mid g) \mid h$

Distributive Laws

$f\ null = null$

$f\ (A, B) = f\,A, f\,B$

$f\ (\S g) = \S y: f\ (\Box g) \cdot \exists x: \Box g \cdot fx{=}y \land gx$

$f\ \textbf{if } b \textbf{ then } x \textbf{ else } y \textbf{ fi} = \textbf{if } b \textbf{ then } f\,x \textbf{ else } f\,y \textbf{ fi}$

$\textbf{if } b \textbf{ then } f \textbf{ else } g \textbf{ fi}\ x = \textbf{if } b \textbf{ then } f\,x \textbf{ else } g\,x \textbf{ fi}$

Arrow Laws

$f: null \rightarrow A$

$A \rightarrow B:\ (A \text{ ‘ } C) \rightarrow (B, D)$

$f: A \rightarrow B \;=\; A: \Box f \land \forall a: A \cdot fa: B$

End of Functions

11.4.8 Quantifiers

Let x be an element, let a, b and c be binary, let n and m be numeric, let f and g be functions, and let p be a predicate.

$\forall v\text{: } null\cdot\ b \ = \ \top$

$\forall v\text{: } x\cdot\ b \ = \ \langle v\text{: } x \to b\rangle\ x$

$\forall v\text{: } A{,}B\cdot\ b \ = \ (\forall v\text{: } A\cdot\ b) \land (\forall v\text{: } B\cdot\ b)$

$\forall v\text{: } (\S v\text{: } D\cdot\ b)\cdot\ c \ = \ \forall v\text{: } D\cdot\ b \Rightarrow c$

$\exists v\text{: } null\cdot\ b \ = \ \bot$

$\exists v\text{: } x\cdot\ b \ = \ \langle v\text{: } x \to b\rangle\ x$

$\exists v\text{: } A{,}B\cdot\ b \ = \ (\exists v\text{: } A\cdot\ b) \lor (\exists v\text{: } B\cdot\ b)$

$\exists v\text{: } (\S v\text{: } D\cdot\ b)\cdot\ c \ = \ \exists v\text{: } D\cdot\ b \land c$

$\Sigma v\text{: } null\cdot\ n \ = \ 0$

$\Sigma v\text{: } x\cdot\ n \ = \ \langle v\text{: } x \to n\rangle\ x$

$(\Sigma v\text{: } A{,}B\cdot\ n) + (\Sigma v\text{: } A\text{‘}B\cdot\ n) \ = \ (\Sigma v\text{: } A\cdot\ n) + (\Sigma v\text{: } B\cdot\ n)$

$\Sigma v\text{: } (\S v\text{: } D\cdot\ b)\cdot\ n \ = \ \Sigma v\text{: } D\cdot\ \textbf{if } b \textbf{ then } n \textbf{ else } 0 \textbf{ fi}$

$\Pi v\text{: } null\cdot\ n \ = \ 1$

$\Pi v\text{: } x\cdot\ n \ = \ \langle v\text{: } x \to n\rangle\ x$

$(\Pi v\text{: } A{,}B\cdot\ n) \times (\Pi v\text{: } A\text{‘}B\cdot\ n) \ = \ (\Pi v\text{: } A\cdot\ n) \times (\Pi v\text{: } B\cdot\ n)$

$\Pi v\text{: } (\S v\text{: } D\cdot\ b)\cdot\ n \ = \ \Pi v\text{: } D\cdot\ \textbf{if } b \textbf{ then } n \textbf{ else } 1 \textbf{ fi}$

$MIN\ v\text{: } null\cdot\ n \ = \ \infty$

$MIN\ v\text{: } x\cdot\ n \ = \ \langle v\text{: } x \to n\rangle\ x$

$MIN\ v\text{: } A{,}B\cdot\ n \ = \ min\ (MIN\ v\text{: } A\cdot\ n)\ (MIN\ v\text{: } B\cdot\ n)$

$MIN\ v\text{: } (\S v\text{: } D\cdot\ b)\cdot\ n \ = \ MIN\ v\text{: } D\cdot\ \textbf{if } b \textbf{ then } n \textbf{ else } \infty \textbf{ fi}$

$MAX\ v\text{: } null\cdot\ n \ = \ -\infty$

$MAX\ v\text{: } x\cdot\ n \ = \ \langle v\text{: } x \to n\rangle\ x$

$MAX\ v\text{: } A{,}B\cdot\ n \ = \ max\ (MAX\ v\text{: } A\cdot\ n)\ (MAX\ v\text{: } B\cdot\ n)$

$MAX\ v\text{:}(\S v\text{: } D\cdot\ b)\cdot\ n \ = \ MAX\ v\text{: } D\cdot\ \textbf{if } b \textbf{ then } n \textbf{ else } -\infty \textbf{ fi}$

$\S v\text{: } null\cdot\ b \ = \ null$

$\S v\text{: } x\cdot\ b \ = \ \textbf{if } \langle v\text{: } x \to b\rangle\ x \textbf{ then } x \textbf{ else } null \textbf{ fi}$

$\S v\text{: } A{,}B\cdot\ b \ = \ (\S v\text{: } A\cdot\ b), (\S v\text{: } B\cdot\ b)$

$\S v\text{: } A\text{‘}B\cdot\ b \ = \ (\S v\text{: } A\cdot\ b)\ \text{‘}\ (\S v\text{: } B\cdot\ b)$

$\S v\text{: } (\S v\text{: } D\cdot\ b)\cdot\ c \ = \ \S v\text{: } D\cdot\ b \land c$

$(MIN\ i\text{: } int\cdot\ i) = -\infty$

$(MAX\ i\text{: } int\cdot\ i) = \infty$

Change of Variable Laws — if d does not appear in b

$\forall r\text{: } fD\cdot\ b \ = \ \forall d\text{: } D\cdot\ \langle r\text{: } fD \to b\rangle\ (fd)$

$\exists r\text{: } fD\cdot\ b \ = \ \exists d\text{: } D\cdot\ \langle r\text{: } fD \to b\rangle\ (fd)$

$MIN\ r\text{: } fD\cdot\ n \ = \ MIN\ d\text{: } D\cdot\ \langle r\text{: } fD \to n\rangle\ (fd)$

$MAX\ r\text{: } fD\cdot\ n \ = \ MAX\ d\text{: } D\cdot\ \langle r\text{: } fD \to n\rangle\ (fd)$

Identity Laws

$\forall v\cdot\ \top$

$\lnot \exists v\cdot\ \bot$

Bunch-Element Conversion Laws

$V\text{: } W \ = \ \forall v\text{: } V\cdot\ \exists w\text{: } W\cdot\ v{=}w$

$fV\text{: } gW \ = \ \forall v\text{: } V\cdot\ \exists w\text{: } W\cdot\ fv{=}gw$

Distributive Laws — if $D \ne null$ and v does not appear in a

$a \land \forall v\text{: } D\cdot\ b \ = \ \forall v\text{: } D\cdot\ a \land b$

$a \land \exists v\text{: } D\cdot\ b \ = \ \exists v\text{: } D\cdot\ a \land b$

$a \lor \forall v\text{: } D\cdot\ b \ = \ \forall v\text{: } D\cdot\ a \lor b$

$a \lor \exists v\text{: } D\cdot\ b \ = \ \exists v\text{: } D\cdot\ a \lor b$

$a \Rightarrow \forall v\text{: } D\cdot\ b \ = \ \forall v\text{: } D\cdot\ a \Rightarrow b$

$a \Rightarrow \exists v\text{: } D\cdot\ b \ = \ \exists v\text{: } D\cdot\ a \Rightarrow b$

Idempotent Laws — if $D \ne null$ and v does not appear in b

$\forall v\text{: } D\cdot\ b \ = \ b$

$\exists v\text{: } D\cdot\ b \ = \ b$

Absorption Laws — if $x\text{: } D$

$\langle v\text{: } D \to b\rangle\ x \land \exists v\text{: } D\cdot\ b \ = \ \langle v\text{: } D \to b\rangle\ x$

$\langle v\text{: } D \to b\rangle\ x \lor \forall v\text{: } D\cdot\ b \ = \ \langle v\text{: } D \to b\rangle\ x$

$\langle v\text{: } D \to b\rangle\ x \land \forall v\text{: } D\cdot\ b \ = \ \forall v\text{: } D\cdot\ b$

$\langle v\text{: } D \to b\rangle\ x \lor \exists v\text{: } D\cdot\ b \ = \ \exists v\text{: } D\cdot\ b$

Antidistributive Laws — if $D \ne null$ and v does not appear in a

$a \Leftarrow \exists v\text{: } D\cdot\ b \ = \ \forall v\text{: } D\cdot\ a \Leftarrow b$

$a \Leftarrow \forall v\text{: } D\cdot\ b \ = \ \exists v\text{: } D\cdot\ a \Leftarrow b$

Specialization Law — if $x: D$

$$\forall v: D\cdot b \implies \langle v: D\to b\rangle\, x$$

One-Point Laws — if $x: D$ and v does not appear in x

$$\begin{gathered}\forall v: D\cdot v{=}x \Rightarrow b \;=\; \langle v: D\to b\rangle\, x \\ \exists v: D\cdot v{=}x \wedge b \;=\; \langle v: D\to b\rangle\, x\end{gathered}$$

Duality Laws

$$\begin{aligned}&\neg\forall v\cdot b \;=\; \exists v\cdot \neg b \quad \text{(deMorgan)}\\ &\neg\exists v\cdot b \;=\; \forall v\cdot \neg b \quad \text{(deMorgan)}\\ &-MAX\ v\cdot n \;=\; MIN\ v\cdot -n\\ &-MIN\ v\cdot n \;=\; MAX\ v\cdot -n\end{aligned}$$

Solution Laws

$$\begin{aligned}&\S v: D\cdot \top \;=\; D\\ &(\S v: D\cdot b):\ D\\ &\S v: D\cdot \bot \;=\; null\\ &(\S v\cdot b): (\S v\cdot c) \;=\; \forall v\cdot b \Rightarrow c\\ &(\S v\cdot b), (\S v\cdot c) \;=\; \S v\cdot b \vee c\\ &(\S v\cdot b)\ `\ (\S v\cdot c) \;=\; \S v\cdot b \wedge c\\ &x: \S p \;=\; x: \square p \ \wedge\ px\\ &\forall f \;=\; (\S f){=}(\square f)\\ &\exists f \;=\; (\S f){\ne}null\end{aligned}$$

Bounding Laws

if v does not appear in n

$$\begin{aligned}&n > (MAX\ v: D\cdot m) \;\Rightarrow\; (\forall v: D\cdot n{>}m)\\ &n < (MIN\ v: D\cdot m) \;\Rightarrow\; (\forall v: D\cdot n{<}m)\\ &n \ge (MAX\ v: D\cdot m) \;=\; (\forall v: D\cdot n{\ge}m)\\ &n \le (MIN\ v: D\cdot m) \;=\; (\forall v: D\cdot n{\le}m)\\ &n \ge (MIN\ v: D\cdot m) \;\Leftarrow\; (\exists v: D\cdot n{\ge}m)\\ &n \le (MAX\ v: D\cdot m) \;\Leftarrow\; (\exists v: D\cdot n{\le}m)\\ &n > (MIN\ v: D\cdot m) \;=\; (\exists v: D\cdot n{>}m)\\ &n < (MAX\ v: D\cdot m) \;=\; (\exists v: D\cdot n{<}m)\end{aligned}$$

Generalization Law — if $x: D$

$$\langle v: D\to b\rangle\, x \implies \exists v: D\cdot b$$

Splitting Laws — for any fixed domain

$$\begin{aligned}&\forall v\cdot a \wedge b \;=\; (\forall v\cdot a) \wedge (\forall v\cdot b)\\ &\exists v\cdot a \wedge b \;\Longrightarrow\; (\exists v\cdot a) \wedge (\exists v\cdot b)\\ &\forall v\cdot a \vee b \;\Longleftarrow\; (\forall v\cdot a) \vee (\forall v\cdot b)\\ &\exists v\cdot a \vee b \;=\; (\exists v\cdot a) \vee (\exists v\cdot b)\\ &\forall v\cdot a \Rightarrow b \;\Longrightarrow\; (\forall v\cdot a) \Rightarrow (\forall v\cdot b)\\ &\forall v\cdot a \Rightarrow b \;\Longrightarrow\; (\exists v\cdot a) \Rightarrow (\exists v\cdot b)\\ &\forall v\cdot a = b \;\Longrightarrow\; (\forall v\cdot a) = (\forall v\cdot b)\\ &\forall v\cdot a = b \;\Longrightarrow\; (\exists v\cdot a) = (\exists v\cdot b)\end{aligned}$$

Commutative Laws

$$\begin{aligned}&\forall v\cdot \forall w\cdot b \;=\; \forall w\cdot \forall v\cdot b\\ &\exists v\cdot \exists w\cdot b \;=\; \exists w\cdot \exists v\cdot b\end{aligned}$$

Semicommutative Laws (Skolem)

$$\begin{aligned}&\exists v\cdot \forall w\cdot b \;\Longrightarrow\; \forall w\cdot \exists v\cdot b\\ &\forall x\cdot \exists y\cdot pxy \;=\; \exists f\cdot \forall x\cdot px(fx)\end{aligned}$$

Domain Change Laws

$$\begin{aligned}&A: B \;\Longrightarrow\; (\forall v: A\cdot b) \Leftarrow (\forall v: B\cdot b)\\ &A: B \;\Longrightarrow\; (\exists v: A\cdot b) \Rightarrow (\exists v: B\cdot b)\\ &\forall v: A\cdot v: B \Rightarrow p \;=\; \forall v: A`B\cdot p\\ &\exists v: A\cdot v: B \wedge p \;=\; \exists v: A`B\cdot p\end{aligned}$$

Extreme Law

$$(MIN\ v\cdot n) \le n \le (MAX\ v\cdot n)$$

Connection Laws (Galois)

$$\begin{aligned}&n{\le}m \;=\; \forall k\cdot\ k{\le}n \Rightarrow k{\le}m\\ &n{\le}m \;=\; \forall k\cdot\ k{<}n \Rightarrow k{<}m\\ &n{\le}m \;=\; \forall k\cdot\ m{\le}k \Rightarrow n{\le}k\\ &n{\le}m \;=\; \forall k\cdot\ m{<}k \Rightarrow n{<}k\end{aligned}$$

Distributive Laws — if $D{\ne}null$ and v does not appear in n

$$\begin{aligned}&max\ n\ (MAX\ v: D\cdot m) \;=\; (MAX\ v: D\cdot max\ n\ m)\\ &max\ n\ (MIN\ v: D\cdot m) \;=\; (MIN\ v: D\cdot max\ n\ m)\\ &min\ n\ (MAX\ v: D\cdot m) \;=\; (MAX\ v: D\cdot min\ n\ m)\\ &min\ n\ (MIN\ v: D\cdot m) \;=\; (MIN\ v: D\cdot min\ n\ m)\\ &n + (MAX\ v: D\cdot m) \;=\; (MAX\ v: D\cdot n{+}m)\\ &n + (MIN\ v: D\cdot m) \;=\; (MIN\ v: D\cdot n{+}m)\\ &n - (MAX\ v: D\cdot m) \;=\; (MIN\ v: D\cdot n{-}m)\\ &n - (MIN\ v: D\cdot m) \;=\; (MAX\ v: D\cdot n{-}m)\\ &(MAX\ v: D\cdot m) - n \;=\; (MAX\ v: D\cdot m{-}n)\\ &(MIN\ v: D\cdot m) - n \;=\; (MIN\ v: D\cdot m{-}n)\\ &n{\ge}0 \;\Rightarrow\; n \times (MAX\ v: D\cdot m) \;=\; (MAX\ v: D\cdot n{\times}m)\\ &n{\ge}0 \;\Rightarrow\; n \times (MIN\ v: D\cdot m) \;=\; (MIN\ v: D\cdot n{\times}m)\\ &n{\le}0 \;\Rightarrow\; n \times (MAX\ v: D\cdot m) \;=\; (MIN\ v: D\cdot n{\times}m)\\ &n{\le}0 \;\Rightarrow\; n \times (MIN\ v: D\cdot m) \;=\; (MAX\ v: D\cdot n{\times}m)\\ &n \times (\Sigma v: D\cdot m) \;=\; (\Sigma v: D\cdot n{\times}m)\\ &(\Pi v: D\cdot m)^n \;=\; (\Pi v: D\cdot m^n)\end{aligned}$$

End of Quantifiers

11.4.9 Limits

$(MAX\ m\cdot\ MIN\ n\cdot\ f(m+n))\ \le\ (LIM\ f)\ \le\ (MIN\ m\cdot\ MAX\ n\cdot\ f(m+n))$
$\exists m\cdot\ \forall n\cdot\ p(m+n)\ \Rightarrow\ LIM\ p\ \Rightarrow\ \forall m\cdot\ \exists n\cdot\ p(m+n)$
$(LIM\ n\cdot\ n) = \infty$

End of Limits

11.4.10 Specifications and Programs

For specifications P , Q , R , and S , and binary b ,

$ok\ =\ x'=x\ \land\ y'=y\ \land\ ...$
$x{:=}\ e\ =\ x'=e\ \land\ y'=y\ \land\ ...$
$P.\ Q\ =\ \exists x'', y'', ...\cdot\ \langle x', y', ...\rightarrow P\rangle\ x''\ y''\ ...\ \land\ \langle x, y, ...\rightarrow Q\rangle\ x''\ y''\ ...$
$P\|Q\ =\ \exists t_P, t_Q\cdot\ \langle t'\rightarrow P\rangle t_P\ \land\ \langle t'\rightarrow Q\rangle t_Q\ \land\ t' = max\ t_P\ t_Q$
if b **then** P **else** Q **fi** $=\ b \land P \lor \neg b \land Q$
var $x{:}\ T\cdot\ P\ =\ \exists x, x'{:}\ T\cdot\ P$
frame $x\cdot\ P\ =\ P\ \land\ y'=y\ \land\ ...$
while b **do** P **od** $=\ t'\ge t \land$ **if** b **then** $P.\ t{:=}\ t+1.$ **while** b **do** P **od else** ok **fi**
$\forall\sigma, \sigma'\cdot$ **if** b **then** $P.\ W$ **else** ok **fi** $\Leftarrow\ W\ \Rightarrow\ \forall\sigma, \sigma'\cdot$ **while** b **do** P **od** $\Leftarrow\ W$
$(Fmn\ \Leftarrow\ m{=}n \land ok) \land (Fik\ \Leftarrow\ m{\le}i{<}j{<}k{\le}n \land (Fij.\ Fjk))$
$\Rightarrow\ Fmn\ \Leftarrow$ **for** $i{:=}\ m;..n$ **do** $m{\le}i{<}n \Rightarrow Fi(i+1)$ **od**
$Im{\Rightarrow}I'n\ \Leftarrow$ **for** $i{:=}\ m;..n$ **do** $m{\le}i{<}n \land Ii \Rightarrow I'(i+1)$ **od**
wait until $w\ =\ t{:=}\ max\ t\ w$
assert b $=$ **if** b **then** ok **else** $print$ “error”. **wait until** ∞ **fi**
ensure $b\ =\ b \land ok$
$P.$ (P **result** e)$=e$ but do not double-prime or substitute in (P **result** e)
$c?\ =\ r{:=}\ r+1$
$c\ =\ \mathbb{M}c_{\ rc-1}$
$c!\ e\ =\ \mathbb{M}c_{\ wc} = e\ \land\ \mathbb{T}c_{\ wc} = t\ \land\ (wc{:=}\ wc+1)$
$\surd c\ =\ \mathbb{T}c_{\ rc}$ + (transit time) $\le t$
ivar $x{:}\ T\cdot\ S\ =\ \exists x{:}\ time\rightarrow T\cdot\ S$
chan $c{:}\ T\cdot\ P$ $=$ $\exists\mathbb{M}c{:}\ \infty{*}T\cdot\ \exists\mathbb{T}c{:}\ \infty{*}xreal\cdot$ **var** rc , $wc{:}\ xnat := 0\cdot\ P$

$ok.\ P\ =\ P.\ ok\ =\ P$	identity
$P.\ (Q.\ R)\ =\ (P.\ Q).\ R$	associativity
$P{\lor}Q.\ R{\lor}S\ =\ (P.\ R) \lor (P.\ S) \lor (Q.\ R) \lor (Q.\ S)$	distributivity
if b **then** P **else** Q **fi**. R $=$ **if** b **then** $P.\ R$ **else** $Q.\ R$ **fi**	distributivity (unprimed b)
$P.$ **if** b **then** Q **else** R **fi** $=$ **if** $P.\ b$ **then** $P.\ Q$ **else** $P.\ R$ **fi**	distributivity (unprimed b)
$P \,\|\, Q\ =\ Q \,\|\, P$	symmetry
$P \,\|\, (Q \,\|\, R)\ =\ (P \,\|\, Q) \,\|\, R$	associativity
$P \,\|\, t'{=}t\ =\ P\ =\ t'{=}t \,\|\, P$	identity
$P \,\|\, Q{\lor}R\ =\ (P \,\|\, Q) \lor (P \,\|\, R)$	distributivity
$P \,\|$ **if** b **then** Q **else** R **fi** $=$ **if** b **then** $P \,\|\, Q$ **else** $P \,\|\, R$ **fi**	distributivity
if b **then** $P\|Q$ **else** $R\|S$ **fi** $=$ **if** b **then** P **else** R **fi** $\|$ **if** b **then** Q **else** S **fi**	distributivity
$x{:=}$ **if** b **then** e **else** f **fi** $=$ **if** b **then** $x{:=}\ e$ **else** $x{:=}\ f$ **fi**	functional-imperative

End of Specifications and Programs

11.4.11 Substitution

Let x and y be different boundary state variables, let e and f be expressions of the prestate, and let P be a specification.

$x{:=}\,e.\,P \;=\;$ (for x substitute e in P)

$(x{:=}\,e \,\|\, y{:=}\,f).\; P \;=\;$ (for x substitute e and independently for y substitute f in P)

End of Substitution

11.4.12 Conditions

Let P and Q be any specifications, and let C be a precondition, and let C' be the corresponding postcondition (in other words, C' is the same as C but with primes on all the state variables).

$C \land (P.\,Q) \;\Leftarrow\; C{\land}P.\,Q$

$C \Rightarrow (P.Q) \;\Leftarrow\; C{\Rightarrow}P.\,Q$

$(P.Q) \land C' \;\Leftarrow\; P.\,Q{\land}C'$

$(P.Q) \Leftarrow C' \;\Leftarrow\; P.\,Q{\Leftarrow}C'$

$P.\,C{\land}Q \;\Leftarrow\; P{\land}C'.\,Q$

$P.\,Q \;\Leftarrow\; P{\land}C'.\; C{\Rightarrow}Q$

C is a sufficient precondition for P to be refined by S
if and only if $C{\Rightarrow}P$ is refined by S .

C' is a sufficient postcondition for P to be refined by S
if and only if $C'{\Rightarrow}P$ is refined by S .

End of Conditions

11.4.13 Refinement

Refinement by Steps (Stepwise Refinement) (monotonicity, transitivity)

If $A \Leftarrow$ **if** b **then** C **else** D **fi** and $C \Leftarrow E$ and $D \Leftarrow F$ are theorems,
then $A \Leftarrow$ **if** b **then** E **else** F **fi** is a theorem.

If $A \Leftarrow B.C$ and $B \Leftarrow D$ and $C \Leftarrow E$ are theorems, then $A \Leftarrow D.E$ is a theorem.

If $A \Leftarrow B\|C$ and $B \Leftarrow D$ and $C \Leftarrow E$ are theorems, then $A \Leftarrow D\|E$ is a theorem.

If $A \Leftarrow B$ and $B \Leftarrow C$ are theorems, then $A \Leftarrow C$ is a theorem.

Refinement by Parts (monotonicity, conflation)

If $A \Leftarrow$ **if** b **then** C **else** D **fi** and $E \Leftarrow$ **if** b **then** F **else** G **fi** are theorems,
then $A{\land}E \Leftarrow$ **if** b **then** $C{\land}F$ **else** $D{\land}G$ **fi** is a theorem.

If $A \Leftarrow B.C$ and $D \Leftarrow E.F$ are theorems, then $A{\land}D \Leftarrow B{\land}E.\,C{\land}F$ is a theorem.

If $A \Leftarrow B\|C$ and $D \Leftarrow E\|F$ are theorems, then $A{\land}D \Leftarrow B{\land}E \,\|\, C{\land}F$ is a theorem.

If $A \Leftarrow B$ and $C \Leftarrow D$ are theorems, then $A{\land}C \Leftarrow B{\land}D$ is a theorem.

Refinement by Cases

$P \Leftarrow$ **if** b **then** Q **else** R **fi** is a theorem if and only if
$P \Leftarrow b \land Q$ and $P \Leftarrow \neg b \land R$ are theorems.

End of Refinement

End of Laws

11.5 Names

abs: $xreal \rightarrow \S r: xreal \cdot r \ge 0$	$abs\ r\ =\ \mathbf{if}\ r \ge 0\ \mathbf{then}\ r\ \mathbf{else}\ -r\ \mathbf{fi}$
bin (the binary values)	$bin\ =\ \top, \bot$
ceil: $real \rightarrow int$	$r \le ceil\ r < r+1$
char (the characters)	$char\ =\ ..., \text{“a”}, \text{“A”}, ...$
div: $real \rightarrow (\S r: real \cdot r>0) \rightarrow int$	$div\ x\ y\ =\ floor\ (x/y)$
divides: $(nat+1) \rightarrow int \rightarrow bin$	$divides\ n\ i\ =\ i/n: int$
entro: $prob \rightarrow \S r: xreal \cdot r \ge 0$	$entro\ p\ =\ p \times info\ p\ +\ (1-p) \times info\ (1-p)$
even: $int \rightarrow bin$	$even\ i\ =\ i/2: int$
	$even\ =\ divides\ 2$
floor: $real \rightarrow int$	$floor\ r \le r < floor\ r + 1$
info: $prob \rightarrow \S r: xreal \cdot r \ge 0$	$info\ p\ =\ -\ log\ p$
int (the integers)	$int\ =\ nat, -nat$
LIM (limit quantifier)	see Laws
log: $(\S r: xreal \cdot r \ge 0) \rightarrow xreal$	$log\ (2^x) = x$
	$log\ (x \times y)\ =\ log\ x + log\ y$
max: $xrat \rightarrow xrat \rightarrow xrat$	$max\ x\ y\ =\ \mathbf{if}\ x \ge y\ \mathbf{then}\ x\ \mathbf{else}\ y\ \mathbf{fi}$
	$-\ max\ a\ b\ =\ min\ (-a)\ (-b)$
MAX (maximum quantifier)	see Laws
min: $xrat \rightarrow xrat \rightarrow xrat$	$min\ x\ y\ =\ \mathbf{if}\ x \le y\ \mathbf{then}\ x\ \mathbf{else}\ y\ \mathbf{fi}$
	$-\ min\ a\ b\ =\ max\ (-a)\ (-b)$
MIN (minimum quantifier)	see Laws
mod: $real \rightarrow (\S r: real \cdot r>0) \rightarrow real$	$0 \le mod\ a\ d < d$
	$a\ =\ div\ a\ d \times d\ +\ mod\ a\ d$
nat (the naturals)	$0, nat+1: nat$
	$0, B+1: B\ \Rightarrow\ nat: B$
nil (the empty string)	$\leftrightarrow nil\ =\ 0$
	$nil; S\ =\ S\ =\ S; nil$
	$nil \le S$
null (the empty bunch)	$¢null\ =\ 0$
	$null, A\ =\ A\ =\ A, null$
	$null: A$
odd: $int \rightarrow bin$	$odd\ i\ =\ \neg\ i/2: int$
	$odd\ =\ \neg even$
ok (the empty program)	$ok\ =\ \sigma'=\sigma$
	$ok.P\ =\ P\ =\ P.ok$
prob (probability)	$prob\ =\ \S r: real \cdot 0 \le r \le 1$
rand (random number)	$rand\ n: 0,..n$
rat (the rationals)	$rat\ =\ int/(nat+1)$
real (the reals)	$r: real\ =\ r: xreal\ \wedge\ -\infty<r<\infty$
suc: $nat \rightarrow (nat+1)$	$suc\ n = n+1$
xint (the extended integers)	$xint\ =\ -\infty, int, \infty$
xnat (the extended naturals)	$xnat\ =\ nat, \infty$
xrat (the extended rationals)	$xrat\ =\ -\infty, rat, \infty$
xreal (the extended reals)	$x: xreal\ =\ \exists f: nat \rightarrow rat \cdot x = LIM\ f$

End of Names

11.6 Symbols

End of Symbols

11.7 Precedence

0	$\top$ $\bot$ ( ) { } [ ] $\langle\ \rangle$ **if fi** **do od** number text name superscript subscript
1	@ juxtaposition
2	prefix– ¢ $ $\leftrightarrow$ # * ~ ↯ □ $\rightarrow$ √
3	× / ∩
4	+ infix– $^+$ ∪
5	; ;.. ‘
6	, ,.. | ◁ ▷
7	= ⧧ < > ≤ ≥ : :: ∈ ⊆
8	¬
9	∧
10	∨
11	⇒ ⇐
12	:= ! ?
13	**exit when** **go to** **wait until** **assert** **ensure** **or**
14	. ‖ **result**
15	∀· ∃· Σ· Π· §· *LIM*· *MAX*· *MIN*· **var**· **ivar**· **chan**· **frame**·
16	**=** **⟹** **⟸**

Superscripting and subscripting serve to bracket all operations within them.

Juxtaposition associates from left to right, so $a\ b\ c$ means the same as $(a\ b)\ c$. The infix operators @ / – associate from left to right. The infix operators * → associate from right to left. The infix operators × ∩ + $^+$ ∪ ; ‘ , | ∧ ∨ . ‖ are associative (they associate in both directions).

On levels 7, 11, and 16 the operators are continuing. For example, $a=b=c$ neither associates to the left nor associates to the right, but means the same as $a=b \wedge b=c$. On any one of these levels, a mixture of continuing operators can be used. For example, $a \le b < c$ means the same as $a \le b \wedge b < c$.

The operators **=** **⟹** **⟸** are identical to = ⇒ ⇐ except for precedence.

End of Precedence

11.8 Distribution

The operators in the following expressions distribute over bunch union in any operand:

$[A]$ $A@B$ $A\ B$ $-A$ $\$A$ $\leftrightarrow A$ $\#A$ $\sim A$

A^B A_B $A \times B$ A/B $A \cap B$ $A+B$ $A-B$ A^+B $A \cup B$ $A;B$ $A`B$

$\neg A$ $A \wedge B$ $A \vee B$

The operator in $A*B$ distributes over bunch union in its left operand only.

End of Distribution

End of Reference

End of a Practical Theory of Programming