FUNCTION THEORY

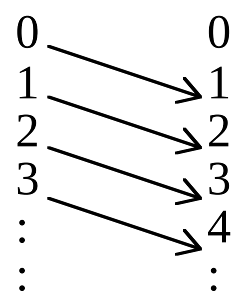
$\lambda v\text{: } D\cdot\ b$ “map v in D to b”

body (may use v)

domain, type (a bunch)

variable, parameter (a fresh name)

$\lambda n\text{: } nat\cdot\ n+1$



Δf “domain of f”

$f\ x$ “f applied to x” or “f of x”

$\Delta\ \lambda n\text{: } nat\cdot\ n+1 \ = \ nat$

$(\lambda n\text{: } nat\cdot\ n+1)\ 3 \ = \ 3+1 \ = \ 4$

<u>Domain Axiom</u>:

$\Delta\ \lambda v\colon D\cdot b\ \ =\ \ D$

<u>Renaming Axiom</u>: If v and w are names, and neither v nor w appears in D , and w does not appear in b , then

$\lambda v\colon D\cdot b\ \ =\ \ \lambda w\colon D\cdot$ (substitute w for v in b)

<u>Application Axiom</u>: If element $x\colon D$, then

$(\lambda v\colon D\cdot b)\ x\ \ =\ $ (substitute x for v in b)

Two variables:

$max\ \ =\ \ \lambda x\colon xrat\cdot \lambda y\colon xrat\cdot$ **if** $x{\ge}y$ **then** x **else** y

$max\ 3\ \ =\ \ \lambda y\colon xrat\cdot$ **if** $3{\ge}y$ **then** 3 **else** y

$max\ 3\ 5\ \ =\ \ 5$

Predicate (function with boolean result):

$even\ \ =\ \ \lambda i\colon int\cdot\ i/2\colon int$

Relation (function with predicate result):

$divides\ \ =\ \ \lambda n\colon nat{+}1\cdot \lambda i\colon int\cdot\ i/n\colon int$

<u>Selective union</u>:

$f \mid g$ "f otherwise g"

$\Delta(f \mid g) = \Delta f, \Delta g$

$(f \mid g)\ x = \textbf{if } x{:}\ \Delta f \textbf{ then } f\ x \textbf{ else } g\ x$

All the rules of proof apply to the body of function $\lambda v{:}\ D\cdot\ b$ with the additional local axiom $v{:}\ D$.

Abbreviated Function Notations

$$\lambda x, y{:}\ xrat\cdot \textbf{ if } x \ge y \textbf{ then } x \textbf{ else } y$$

$$= \lambda x{:}\ xrat\cdot\ \lambda y{:}\ xrat\cdot \textbf{ if } x \ge y \textbf{ then } x \textbf{ else } y$$

$$\lambda n\cdot\ n+1 = \lambda n{:}\ nat\cdot\ n+1$$

$$2 \rightarrow 3 = \lambda n{:}\ 2\cdot\ 3$$

$x+3$ might be $\lambda x{:}\ int\cdot\ \lambda y{:}\ int\cdot\ x+3$

Scope and Substitution

<u>local</u> (bound, hidden, private):

introduction is inside the expression (formal)

<u>nonlocal</u> (free, global, visible, public):

introduction is outside the expression (formal or informal)

$$(\lambda x\cdot \quad x \quad (\lambda x\cdot \quad x \quad) \quad x \quad) \; 3$$

$$= \quad (\quad 3 \quad (\lambda x\cdot \quad x \quad) \quad 3 \quad)$$

$$(\lambda y\cdot \quad x \quad y \quad (\lambda x\cdot \quad x \quad y \quad) \quad x \quad y \quad) \; x$$

$$= \quad (\lambda y\cdot \quad x \quad y \quad (\lambda z\cdot \quad z \quad y \quad) \quad x \quad y \quad) \; x$$

$$= \quad (\quad x \quad x \quad (\lambda z\cdot \quad z \quad x \quad) \quad x \quad x \quad)$$

QUANTIFIERS

Any binary symmetric associative operator defined over the range of a function can be used to define a quantifier.

$\forall p$ is defined from $\wedge$

$\exists p$ is defined from $\vee$

Σf is defined from $+$

Πf is defined from $\times$

$\forall \lambda r\text{: } rat\cdot\ r<0 \vee r=0 \vee r>0$

$\exists \lambda n\text{: } nat\cdot\ n=0$

$\Sigma \lambda n\text{: } nat+1\cdot\ 1/2^n$

$\Pi \lambda n\text{: } nat+1\cdot\ (4n^2)/(4n^2-1)$

Abbreviated Quantifier Notations

$\forall r\text{: } rat\cdot\ r<0 \vee r=0 \vee r>0$

$\Sigma n\text{: } nat+1\cdot\ 1/2^n$

$$\forall x, y\text{: } rat\cdot\ x = y+1 \ \Rightarrow\ x > y$$

$$=\quad \forall x\text{: } rat\cdot\ \forall y\text{: } rat\cdot\ x = y+1 \ \Rightarrow\ x > y$$

$$\Sigma n, m\text{: } 0,..10\cdot\ n\times m$$

$$=\quad \Sigma n\text{: } 0,..10\cdot\ \Sigma m\text{: } 0,..10\cdot\ n\times m$$

$$\forall v\colon null\cdot b \;=\; \top$$

$$\forall v\colon x\cdot b \;=\; (\lambda v\colon x\cdot b)\; x$$

$$\forall v\colon A{,}B\cdot b \;=\; (\forall v\colon A\cdot b) \wedge (\forall v\colon B\cdot b)$$

$$\exists v\colon null\cdot b \;=\; \bot$$

$$\exists v\colon x\cdot b \;=\; (\lambda v\colon x\cdot b)\; x$$

$$\exists v\colon A{,}B\cdot b \;=\; (\exists v\colon A\cdot b) \vee (\exists v\colon B\cdot b)$$

$$\Sigma v\colon null\cdot n \;=\; 0$$

$$\Sigma v\colon x\cdot n \;=\; (\lambda v\colon x\cdot n)\; x$$

$$(\Sigma v\colon A{,}B\cdot n) + (\Sigma v\colon A\text{‘}B\cdot n) \;=\; (\Sigma v\colon A\cdot n) + (\Sigma v\colon B\cdot n)$$

$$\Pi v\colon null\cdot n \;=\; 1$$

$$\Pi v\colon x\cdot n \;=\; (\lambda v\colon x\cdot n)\; x$$

$$(\Pi v\colon A{,}B\cdot n) \times (\Pi v\colon A\text{‘}B\cdot n) \;=\; (\Pi v\colon A\cdot n) \times (\Pi v\colon B\cdot n)$$

$MAX\ x{:}\ rat\cdot 4{\times}x - x^2 \ = \ 4$

$MAX\ v{:}\ null\cdot n \ = \ -\infty$

$MAX\ v{:}\ x\cdot n \ = \ (\lambda v{:}\ x\cdot n)\ x$

$MAX\ v{:}\ A{,}B\cdot n \ = \ max\ (MAX\ v{:}\ A\cdot n)\ (MAX\ v{:}\ B\cdot n)$

Solution Quantifier

§p is the (bunch of) solutions of predicate p

§$v{:}\ null\cdot b \ = \ null$

§$v{:}\ x\cdot b \ = \ \textbf{if}\ (\lambda v{:}\ x\cdot b)\ x\ \textbf{then}\ x\ \textbf{else}\ null$

§$v{:}\ A{,}B\cdot b \ = \ ($§$v{:}\ A\cdot b),\ ($§$v{:}\ B\cdot b)$

§$i{:}\ int\cdot i^2{=}4 \ = \ -2,\ 2$

§$n{:}\ nat\cdot n{<}3 \ = \ 0,..3$

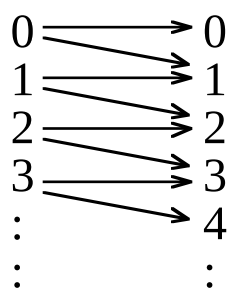
partial: sometimes no result

total: always at least one result

deterministic: always at most one result

nondeterministic: sometimes more than one result

$\lambda n\text{: } nat\cdot\ n,\ n+1$ total and nondeterministic



$(\lambda n\text{: } nat\cdot\ n,\ n+1)\ 3 \quad = \quad 3,\ 4$

$\lambda v\text{: } D\cdot\ A,\ B \quad = \quad (\lambda v\text{: } D\cdot\ A),\ (\lambda v\text{: } D\cdot\ B)$

$(f,\ g)\ x \quad = \quad fx,\ gx$

distribution

$double = \lambda n: nat \cdot n+n$

$double\ 2 = 4$

$double\ (2, 3) = double\ 2, double\ 3 = 4, 6$

$double\ (2, 3) \neq (2, 3) + (2, 3) = 4, 5, 6$

$tiny = \lambda S: 2\{nat\} \cdot \$S < 3$

$tiny\ \{null\} = \top$

$tiny\ \{0, 1, 2, 3\} = \bot$

$tiny\ null = null$

Function Inclusion:

$$f: g \;=\; \Delta g: \Delta f \;\wedge\; \forall x: \Delta g \cdot fx: gx$$

$$f: A \rightarrow B \;=\; A: \Delta f \;\wedge\; \forall a: A \cdot fa: B$$

Function Equality:

$$f = g \;=\; \Delta f = \Delta g \;\wedge\; \forall x: \Delta f \cdot fx = gx$$

$$suc \;=\; \lambda n: nat \cdot n+1$$

$$nat \rightarrow nat \;=\; \lambda n: nat \cdot nat$$

$$suc: nat \rightarrow nat$$

$$= \quad nat: nat \;\wedge\; \forall n: nat \cdot suc\ n: nat$$

$$= \quad \forall n: nat \cdot n+1: nat$$

$$even: int \rightarrow bool$$

$$max: rat \rightarrow rat \rightarrow rat$$

If $A: B$ and $f: B \to C$ and $C: D$ then $f: A \to D$.

$$suc: 0,..10 \to int$$

The function

$$\lambda f: 0,..10 \to int \cdot \forall n: 0,..10 \cdot even\ (f\ n)$$

can be applied to *suc* .

Function Composition

Let f and g be functions such that $\neg f: \Delta g$. Then $g\,f$ is the <u>composition</u> of g and f .

$\Delta(g\,f) \;=\; \S x\colon \Delta f \cdot f x\colon \Delta g$

$(g\,f)\,x \;=\; g\,(f\,x)$

$$
\begin{aligned}
& \Delta(even\ suc) \\
= \;& \S x\colon \Delta suc \cdot suc\ x\colon \Delta even \\
= \;& \S x\colon nat \cdot x{+}1\colon int \\
= \;& nat
\end{aligned}
$$

$(even\ suc)\ 3 \;=\; even\ (suc\ 3) \;=\; even\ 4 \;=\; \top$

$(-suc)\ 3 \;=\; -(suc\ 3) \;=\; -4$

$(\neg even)\ 3 \;=\; \neg(even\ 3) \;=\; \neg\bot \;=\; \top$

Suppose x and y are not functions, f and g are functions of 1 variable, and h is a function of 2 variables. Then

$$
\begin{aligned}
& h\ f\ x\ g\ y \\
= \quad & (((h\ f)\ x)\ g)\ y \\
= \quad & ((h\ (f\ x))\ g)\ y \\
= \quad & (h\ (f\ x))\ (g\ y) \\
= \quad & h\ (f\ x)\ (g\ y)
\end{aligned}
$$

List as Function

L is like $\lambda n\colon 0,..\#L\cdot\ Ln$

domain of L is $0,..\#L$

$L\ n$ indexing is like application

$L\ M$ list composition is like function composition

$suc\ [3;\ 5;\ 2]\ =\ [4;\ 6;\ 3]$

$1 \rightarrow 21 \mid [10;\ 11;\ 12]\ =\ [10;\ 21;\ 12]$

$\Sigma L\ =\ \Sigma n\colon 0,..\#L\cdot\ Ln$

Limits

$f: nat \to rat$

$f0\ f1\ f2\ ...$ is a sequence of rationals

$$(MAX\ m\cdot\ MIN\ n\cdot\ f(m+n)) \ \le\ (LIM\ f) \ \le\ (MIN\ m\cdot\ MAX\ n\cdot\ f(m+n))$$

$LIM\ n\cdot\ 1/(n+1) \ =\ 0$

$-1 \le (LIM\ n\cdot\ (-1)^n) \le 1$

$(MIN\ f) \le (LIM\ f) \le (MAX\ f)$

$p: nat \to bool$

$p0\ p1\ p2\ ...$ is a sequence of booleans

$\exists m\cdot\ \forall n\cdot\ p(m+n) \ \Longrightarrow\ LIM\ p \ \Longrightarrow\ \forall m\cdot\ \exists n\cdot\ p(m+n)$

$\exists m\cdot\ \forall n\cdot\ n \ge m \Rightarrow pn \ \Longrightarrow\ LIM\ p$

$\exists m\cdot\ \forall n\cdot\ n \ge m \Rightarrow \neg pn \ \Longrightarrow\ \neg\ LIM\ p$

$\neg\ LIM\ n\cdot\ 1/(n+1)=0$