BOOLEAN THEORY

(logic)

boolean expressions:

- can be used to represent statements about the world (natural or constructed, real or imaginary)
- can be used to represent digital circuits

theorems:

represent true statements

represent circuits with high voltage output

antitheorems:

represent false statements

represent circuits with low voltage output boolean expressions

$\top \quad \bot$

$\neg x$

$x \wedge y \quad x \vee y \quad x \Rightarrow y \quad x \Leftarrow y \quad x = y \quad x \neq y$

if x **then** y **else** z

precedence and parentheses

associative operators: $\wedge \quad \vee \quad = \quad \neq$

$x \wedge y \wedge z$ means either $(x \wedge y) \wedge z$ or $x \wedge (y \wedge z)$

$x \vee y \vee z$ means either $(x \vee y) \vee z$ or $x \vee (y \vee z)$

continuing operators: $\Rightarrow \quad \Leftarrow \quad = \quad \neq$

$x = y = z$ means $x = y \wedge y = z$

$x \Rightarrow y \Rightarrow z$ means $(x \Rightarrow y) \wedge (y \Rightarrow z)$

big operators: $= \quad \Longrightarrow \quad \Longleftarrow$

same as $= \ \Rightarrow \ \Leftarrow$ but later precedence

$x = y \Longrightarrow z$ means $(x = y) \wedge (y \Longrightarrow z)$

1.2

truth tables

	⊤	⊥
¬	⊥	⊤

	⊤⊤	⊤⊥	⊥⊤	⊥⊥
∧	⊤	⊥	⊥	⊥
∨	⊤	⊤	⊤	⊥
⇒	⊤	⊥	⊤	⊤
⇐	⊤	⊤	⊥	⊤
=	⊤	⊥	⊥	⊤
⧧	⊥	⊤	⊤	⊥

	⊤⊤⊤	⊤⊤⊥	⊤⊥⊤	⊤⊥⊥	⊥⊤⊤	⊥⊤⊥	⊥⊥⊤	⊥⊥⊥
if then else	⊤	⊤	⊥	⊥	⊤	⊥	⊤	⊥

variables are for substitution (instantiation)

- add parentheses to maintain precedence

  in $x \land y$ replace x by $\bot$ and y by $\bot \lor \top$

  result: $\bot \land (\bot \lor \top)$

- every occurrence of a variable must be replaced by the same expression

  in $x \land x$ replace x by $\bot$

  result: $\bot \land \bot$

- different variables can be replaced by the same expression

  in $x \land y$ replace x by $\bot$ and y by $\bot$

  result: $\bot \land \bot$

new boolean expressions

(the grass is green)

(the sky is green)

(there is life elsewhere in the universe)

(intelligent messages are coming from space)

1 + 1 = 2

0 / 0 = 5

consistent: no boolean expression is both a theorem and an antitheorem (no overclassified expressions)

complete: every fully instantiated boolean expression is either a theorem or an antitheorem (no unclassified expressions)

Proof Rules

Axiom Rule If a boolean expression is an axiom, then it is a theorem. If a boolean expression is an antiaxiom, then it is an antitheorem.

axiom: $\top$

antiaxiom: $\bot$

axiom: (the grass is green)

antiaxiom: (the sky is green)

axiom: (intelligent messages are coming from space)

$\Rightarrow$ (there is life elsewhere in the universe)

Evaluation Rule If all the boolean subexpressions of a boolean expression are classified, then it is classified according to the truth tables.

<u>Completion Rule</u> If a boolean expression contains unclassified boolean subexpressions, and all ways of classifying them place it in the same class, then it is in that class.

theorem: (there is life elsewhere in the universe) $\lor$ ⊤

theorem: (there is life elsewhere in the universe)
$\lor$ ¬(there is life elsewhere in the universe)

antitheorem: (there is life elsewhere in the universe)
$\land$ ¬(there is life elsewhere in the universe)

Consistency Rule If a classified boolean expression contains boolean subexpressions, and only one way of classifying them is consistent, then they are classified that way.

Example (modus ponens, detachment):

Suppose x and $x \Rightarrow y$ are theorems. If y were an antitheorem, then by the Evaluation Rule, $x \Rightarrow y$ would be an antitheorem. That would be inconsistent. So y is a theorem.

Example (abnegation):

Suppose $\neg x$ is a theorem. If x were a theorem, then by the Evaluation Rule, $\neg x$ would be an antitheorem. That would be inconsistent. So x is an antitheorem.

No need to talk about antiaxioms and antitheorems.

<u>Instance Rule</u> If a boolean expression is classified, then all its instances have that same classification.

axiom: $x = x$

theorem: $x = x$

theorem: $\top = \bot \lor \bot \;\; = \;\; \top = \bot \lor \bot$

theorem: (intelligent messages are coming from space)

= (intelligent messages are coming from space)

Classical Logic: all five rules

Constructive Logic: not Completion Rule

Evaluation Logic: neither Consistency Rule nor Completion Rule

Expression and Proof Format

$a{\wedge}b \;\vee\; c$ not $a \;\wedge\; b{\vee}c$

(	*first part*
$\wedge$	*second part*)

	first part
=	*second part*

	$a \wedge b \Rightarrow c$	Material Implication
$=$	$\neg(a \wedge b) \vee c$	Duality
$=$	$\neg a \vee \neg b \vee c$	Material Implication
$=$	$a \Rightarrow \neg b \vee c$	Material Implication
$=$	$a \Rightarrow (b \Rightarrow c)$	

	$(a \wedge b \Rightarrow c \;=\; a \Rightarrow (b \Rightarrow c))$	Material Imp 3 times
$=$	$(\neg(a \wedge b) \vee c \;=\; \neg a \vee (\neg b \vee c))$	Duality
$=$	$(\neg a \vee \neg b \vee c \;=\; \neg a \vee \neg b \vee c)$	Reflexivity of $=$
$=$	$\top$	

Monotonicity and Antimonotonicity

$\neg a$ is antimonotonic in a

$a \wedge b$ is monotonic in a and monotonic in b

$a \vee b$ is monotonic in a and monotonic in b

$a \Rightarrow b$ is antimonotonic in a and monotonic in b

$a \Leftarrow b$ is monotonic in a and antimonotonic in b

if a **then** b **else** c is monotonic in b and monotonic in c

	$\neg(a \wedge \neg(\underline{a \vee b}))$	use the Law of Generalization
$\Leftarrow$	$\neg(a \wedge \neg a)$	now use the Law of Noncontradiction
$=$	$\top$	

Context

In $a \wedge b$, when changing a , we can assume b .

In $a \wedge b$, when changing b , we can assume a .

In $a \vee b$, when changing a , we can assume $\neg b$.

In $a \vee b$, when changing b , we can assume $\neg a$.

In $a \Rightarrow b$, when changing a , we can assume $\neg b$.

In $a \Rightarrow b$, when changing b , we can assume a .

In $a \Leftarrow b$, when changing a , we can assume b .

In $a \Leftarrow b$, when changing b , we can assume $\neg a$.

In **if** a **then** b **else** c , when changing b , we can assume a .

In **if** a **then** b **else** c , when changing c , we can assume $\neg a$.

	$\neg(a \wedge \neg(a \vee b))$	assume a to simplify $\neg(a \vee b)$
$=$	$\neg(a \wedge \neg(\top \vee b))$	Symmetry Law and Base Law for $\vee$
$=$	$\neg(a \wedge \neg\top)$	Truth Table for $\neg$
$=$	$\neg(a \wedge \bot)$	Base Law for $\wedge$
$=$	$\neg\bot$	Boolean Axiom, or Truth Table for $\neg$
$=$	$\top$	

NUMBER THEORY

(arithmetic)

number expressions can be used to represent quantity

number expressions

$0 \quad 1 \quad 2 \quad 597 \quad 1.2 \quad 1e10 \quad \infty$

$+x \quad -x \quad x+y \quad x-y \quad x\times y \quad x/y \quad \frac{x}{y}$

x^y **if** *a* **then** *x* **else** *y*

boolean expressions

$x=y \quad x \neq y \quad x<y \quad x>y \quad x\le y \quad x\ge y$

CHARACTER THEORY

\`A \`a \` \`\`

succ *pred* **if then else**

$= \quad \neq \quad < \quad > \quad \le \quad \ge$