boolean expressions: represent anything that comes in two kinds

represent statements about the world (natural or constructed, real or imaginary)

represent digital circuits

represent human behavior

theorems: represent one kind

represent true statements

represent circuits with high voltage output

represent innocent behavior

antitheorems: represent the other kind

represent false statements

represent circuits with low voltage output

represent guilty behavior

0 operands $\top \quad \bot$

1 operand $\neg x$

2 operands $x \wedge y \quad x \vee y \quad x \Rightarrow y \quad x \Leftarrow y \quad x = y \quad x \ne y$

3 operands **if** x **then** y **else** z

precedence and parentheses

associative operators: $\wedge \quad \vee \quad = \quad \ne$

$x \wedge y \wedge z$ means either $(x \wedge y) \wedge z$ or $x \wedge (y \wedge z)$

$x \vee y \vee z$ means either $(x \vee y) \vee z$ or $x \vee (y \vee z)$

continuing operators: $\Rightarrow \quad \Leftarrow \quad = \quad \ne$

$x = y = z$ means $x = y \;\wedge\; y = z$

$x \Rightarrow y \Rightarrow z$ means $(x \Rightarrow y) \;\wedge\; (y \Rightarrow z)$

big operators: $\boldsymbol{=} \quad \boldsymbol{\Longrightarrow} \quad \boldsymbol{\Longleftarrow}$

same as $= \quad \Rightarrow \quad \Leftarrow$ but later precedence

$x \boldsymbol{=} y \boldsymbol{\Longrightarrow} z$ means $(x \boldsymbol{=} y) \;\wedge\; (y \boldsymbol{\Longrightarrow} z)$

truth tables

	$\top$	$\bot$
$\neg$	$\bot$	$\top$

	$\top\top$	$\top\bot$	$\bot\top$	$\bot\bot$
$\wedge$	$\top$	$\bot$	$\bot$	$\bot$
$\vee$	$\top$	$\top$	$\top$	$\bot$
$\Longrightarrow$	$\top$	$\bot$	$\top$	$\top$
$\Longleftarrow$	$\top$	$\top$	$\bot$	$\top$
$=$	$\top$	$\bot$	$\bot$	$\top$
$\neq$	$\bot$	$\top$	$\top$	$\bot$

	$\top\top\top$	$\top\top\bot$	$\top\bot\top$	$\top\bot\bot$	$\bot\top\top$	$\bot\top\bot$	$\bot\bot\top$	$\bot\bot\bot$
if then else	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$

variables are for substitution (instantiation)

- add parentheses to maintain precedence

  in $x \wedge y$ replace x by $\bot$ and y by $\bot \vee \top$ result: $\bot \wedge (\bot \vee \top)$

- every occurrence of a variable must be replaced by the same expression

  in $x \wedge x$ replace x by $\bot$ result: $\bot \wedge \bot$

- different variables can be replaced by the same expression or different expressions

  in $x \wedge y$ replace x by $\bot$ and y by $\bot$ result: $\bot \wedge \bot$

  in $x \wedge y$ replace x by $\top$ and y by $\bot$ result: $\top \wedge \bot$

new boolean expressions

(the grass is green)

(the sky is green)

(there is life elsewhere in the universe)

(intelligent messages are coming from space)

$1 + 1 = 2$

$0 / 0 = 5$

---

consistent: no boolean expression is both a theorem and an antitheorem

(no overclassified expressions)

complete: every fully instantiated boolean expression is either a theorem or an antitheorem

(no unclassified expressions)

Proof Rules

Axiom Rule If a boolean expression is an axiom, then it is a theorem.

If a boolean expression is an antiaxiom, then it is an antitheorem.

axiom: $\top$

antiaxiom: $\bot$

axiom: (the grass is green)

antiaxiom: (the sky is green)

axiom: (intelligent messages are coming from space)

$\Rightarrow$ (there is life elsewhere in the universe)

Evaluation Rule If all the boolean subexpressions of a boolean expression are classified,

then it is classified according to the truth tables.

Proof Rules

Completion Rule If a boolean expression contains unclassified boolean subexpressions, and all ways of classifying them place it in the same class, then it is in that class.

theorem: (there is life elsewhere in the universe) $\lor$ $\top$

theorem:
(there is life elsewhere in the universe)
$\lor$ $\neg$(there is life elsewhere in the universe)

antitheorem:
(there is life elsewhere in the universe)
$\land$ $\neg$(there is life elsewhere in the universe)

Proof Rules

Consistency Rule If a classified boolean expression contains boolean subexpressions, and only one way of classifying them is consistent, then they are classified that way.

We are given that x and $x \Rightarrow y$ are theorems. What is y ?

If y were an antitheorem, then by the Evaluation Rule, $x \Rightarrow y$ would be an antitheorem.

That would be inconsistent. So y is a theorem.

We are given that $\neg x$ is a theorem. What is x ?

If x were a theorem, then by the Evaluation Rule, $\neg x$ would be an antitheorem.

That would be inconsistent. So x is an antitheorem.

No need to talk about antiaxioms and antitheorems.

Proof Rules

Instance Rule If a boolean expression is classified,

then all its instances have that same classification.

axiom: $x = x$

theorem: $x = x$

theorem: $\top = \bot \lor \bot \quad = \quad \top = \bot \lor \bot$

theorem: (intelligent messages are coming from space)

$=$ (intelligent messages are coming from space)

Classical Logic: all five rules

Constructive Logic: not Completion Rule

Evaluation Logic: neither Consistency Rule nor Completion Rule

Expression and Proof Format

$a{\wedge}b \;\vee\; c$ **NOT** $a \;\wedge\; b{\vee}c$

(*first part*

∧ *second part*)

C and Java convention

```
while (something) {
        various lines
        in the body
        of the loop
}
```

Expression and Proof Format

$a \land b \lor c$ **NOT** $a \land b \lor c$

$($ *first part*
$\land$ *second part* $)$

first part
$=$ *second part*

	expression0			*expression0=expression1*
$=$	*expression1*	means	$\land$	*expression1=expression2*
$=$	*expression2*		$\land$	*expression2=expression3*
$=$	*expression3*			

Expression and Proof Format

$a \wedge b \;\vee\; c$ **NOT** $a \;\wedge\; b \vee c$

$$
\begin{array}{ll}
(& \textit{first part} \\
\wedge & \textit{second part} \quad)
\end{array}
$$

$$
\begin{array}{ll}
 & \textit{first part} \\
= & \textit{second part}
\end{array}
$$

	expression0	hint0
=	*expression1*	hint1
=	*expression2*	hint2
=	*expression3*	

Expression and Proof Format

Prove $a \wedge b \Rightarrow c \;=\; a \Rightarrow (b \Rightarrow c)$

$$
\begin{aligned}
& a \wedge b \Rightarrow c && \text{Material Implication} \\
= \;& \neg(a \wedge b) \vee c && \text{Duality} \\
= \;& \neg a \vee \neg b \vee c && \text{Material Implication} \\
= \;& a \Rightarrow \neg b \vee c && \text{Material Implication} \\
= \;& a \Rightarrow (b \Rightarrow c) &&
\end{aligned}
$$

Material Implication: $a \Rightarrow c = \neg a \vee b$

Instance of Material Implication: $a \wedge b \Rightarrow c = \neg(a \wedge b) \vee c$

Expression and Proof Format

Prove $a \wedge b \Rightarrow c \;=\; a \Rightarrow (b \Rightarrow c)$

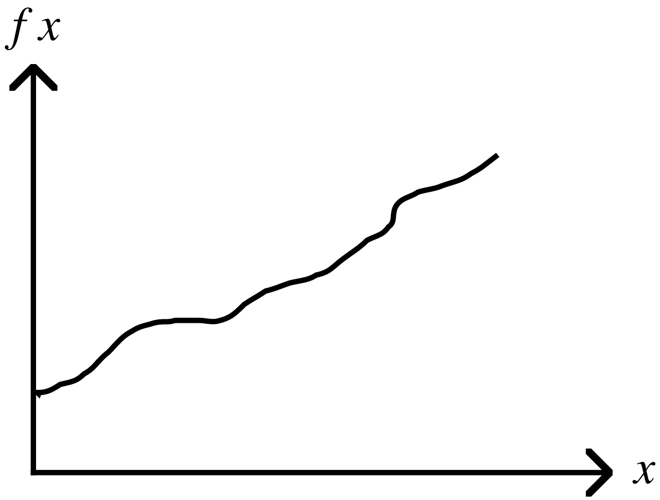
	$a \wedge b \Rightarrow c$	Material Implication
$=$	$\neg(a \wedge b) \vee c$	Duality
$=$	$\neg a \vee \neg b \vee c$	Material Implication
$=$	$a \Rightarrow \neg b \vee c$	Material Implication
$=$	$a \Rightarrow (b \Rightarrow c)$	

	$(a \wedge b \Rightarrow c \;=\; a \Rightarrow (b \Rightarrow c))$	Material Implication 3 times
$=$	$(\neg(a \wedge b) \vee c \;=\; \neg a \vee (\neg b \vee c))$	Duality
$=$	$(\neg a \vee \neg b \vee c \;=\; \neg a \vee \neg b \vee c)$	Reflexivity of $=$
$=$	$\top$	

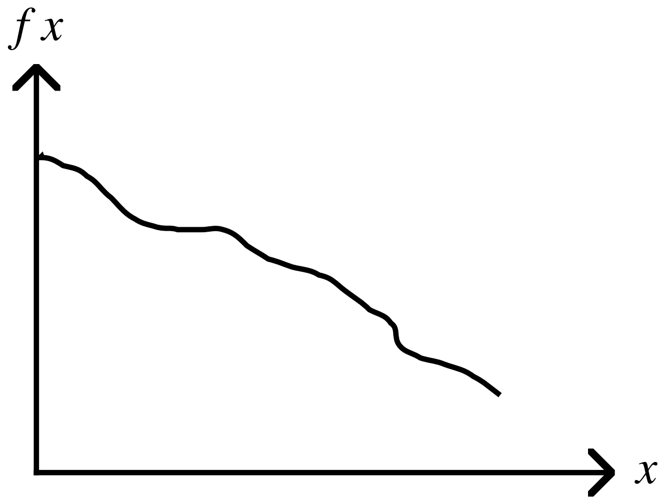
Monotonicity and Antimonotonicity

covariance	**and**	**contravariance**
varies directly as	**and**	**varies inversely as**
nondecreasing	**and**	**nonincreasing**
sorted	**and**	**sorted backwards**

$x \le y \;\Rightarrow\; f\,x \le f\,y$



$x \le y \;\Rightarrow\; f\,x \ge f\,y$



Monotonicity and Antimonotonicity

numbers: $x \le y$ x is less than or equal to y

booleans: $x \Rightarrow y$ x implies y x is falser than or equal to y

Monotonicity and Antimonotonicity

numbers:	$x \le y$	x is less than or equal to y
	$-\infty \le +\infty \qquad 0 \le 1$	smaller $\le$ larger
	$x \le y \Rightarrow f\,x \le f\,y$	f is monotonic
		as x gets larger, $f\,x$ gets larger (or equal)
	$x \le y \Rightarrow f\,x \ge f\,y$	f is antimonotonic
		as x gets larger, $f\,x$ gets smaller (or equal)
booleans:	$x \Rightarrow y$	x implies y $\qquad$ x is stronger than or equal to y
	$\bot \Rightarrow \top$	stronger $\Rightarrow$ weaker
	$x \Rightarrow y \implies f\,x \Rightarrow f\,y$	f is monotonic
		as x gets weaker, $f\,x$ gets weaker (or equal)
	$x \Rightarrow y \implies f\,x \Leftarrow f\,y$	f is antimonotonic
		as x gets weaker, $f\,x$ gets stronger (or equal)

Monotonicity and Antimonotonicity

$\neg a$	antimonotonic in a	
$a \wedge b$	monotonic in a	monotonic in b
$a \vee b$	monotonic in a	monotonic in b
$a \Rightarrow b$	antimonotonic in a	monotonic in b
$a \Leftarrow b$	monotonic in a	antimonotonic in b
if a **then** b **else** c	monotonic in b	monotonic in c

	$\neg(a \wedge \neg(a \vee b))$	use the Law of Generalization $a \Rightarrow a \vee b$
$\Leftarrow$	$\neg(a \wedge \neg a)$	now use the Law of Noncontradiction
$=$	$\top$	

Context

In $a \wedge b$, when changing a , we can assume b .

$$
\begin{array}{ll}
 & a \wedge b \\
 & \downarrow \\
= & c \wedge b
\end{array}
$$

If b is $\top$, we have assumed correctly.

If b is $\bot$, then $a \wedge b$ and $c \wedge b$ are both $\bot$, so the equation is $\top$ anyway.

Context

In $a \wedge b$, when changing a , we can assume b .

In $a \wedge b$, when changing b , we can assume a .

	$\neg(a \wedge \underline{\neg(a \vee b)})$	assume a to simplify $\neg(a \vee b)$
=	$\neg(a \wedge \neg(\top \vee b))$	Symmetry Law and Base Law for $\vee$
=	$\neg(a \wedge \neg\top)$	Truth Table for $\neg$
=	$\neg(a \wedge \bot)$	Base Law for $\wedge$
=	$\neg\bot$	Boolean Axiom, or Truth Table for $\neg$
=	$\top$	

Context

In $a \wedge b$, when changing a , we can assume b .

In $a \wedge b$, when changing b , we can assume a .

In $a \vee b$, when changing a , we can assume $\neg b$.

In $a \vee b$, when changing b , we can assume $\neg a$.

In $a \Rightarrow b$, when changing a , we can assume $\neg b$.

In $a \Rightarrow b$, when changing b , we can assume a .

In $a \Leftarrow b$, when changing a , we can assume b .

In $a \Leftarrow b$, when changing b , we can assume $\neg a$.

In **if** a **then** b **else** c , when changing a , we can assume $b ⧧ c$.

In **if** a **then** b **else** c , when changing b , we can assume a .

In **if** a **then** b **else** c , when changing c , we can assume $\neg a$.

Number Theory

number expressions represent quantity

number expressions

0 1 2 597 1.2 1e10 ∞

$+x$ $-x$ $x+y$ $x-y$ $x \times y$ x/y

x^y **if** a **then** x **else** y

boolean expressions

$x=y$ $x \neq y$ $x<y$ $x>y$ $x \leq y$ $x \geq y$

Character Theory

`A `a ` ``

succ *pred* **if then else**

$= \quad \neq \quad < \quad > \quad \leq \quad \geq$