the Gambler and the Mathematician

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A gambler walked up to a mathematician and said “Let's make a bet. You give me $100 now, and if a certain event happens within a year I'll give you $500,000. I'm betting that it won't happen. Are you willing to bet that it will?”. The mathematician was intrigued; losing $100 wouldn't hurt much, and gaining $500,000 would be great. “What event?”, she asked. She had three purposes in asking this question. First, she needed to be sure that it's an event the gambler cannot prevent. Second, its occurrence has to be clear and knowable to both of them, so they will agree on whether it has occurred. And finally, she needed to know its probability. If its probability is better than 1 in 5,000 then it's a good bet for her, and she should agree to it. If its probability is worse than 1 in 5,000 then it's a bad bet for her, and she should not agree to it.

A standard choice of event used by lotteries is that a particular sequence of decimal digits, chosen in advance by the player, is displayed when a sequence of balls drops, each one displaying one digit. (There may be a variety of winning sequences or partial sequences, but let's keep it simple.) This event cannot be influenced by anyone, everyone can see what the winning sequence is, and it's easy to calculate the probability. If there are 3 or fewer balls in the sequence, the probability is 1 in 1,000 or better, and the mathematician should play. If there are 4 or more balls in the sequence, the probability is 1 in 10,000 or worse, and the mathematician should not play. But this is not the event that the gambler had in mind.

The gambler said “I'll give you a choice of events. If you like, the event can be that your house burns down sometime in the coming year.”. The mathematician thought “That's a gruesome, unpleasant choice of event, but that's not really relevant to whether I should bet. The gambler cannot prevent it, and it's clear whether it has happened. I just need to know its probability.”. The gambler continued: “I have collected all the fire statistics for your neighborhood, and the probability that your house will burn down next year is 1 in 10,000.”. The mathematician immediately saw that this probability does not give her a fair chance of winning, so she declined to bet. “What's my other choice?”, she asked. “The other choice is that you win the Field's medal.” That's one of the highest honors for a mathematician, a delightfully pleasant choice of event, but again, the pleasantness of the event is irrelevant. She knew, and apparently the gambler did not know, that she was among the best thousand mathematicians in the world. Not knowing where among the best thousand mathematicians she stood, she estimated her chance at about 1 in 1,000, and agreed to the bet.

A mathematician is different from ordinary people; ordinary people do not understand mathematics. An ordinary person usually agrees to bet that their house will burn down, even when it's a bad bet for them. The $500,000 is called insurance, the gambler is called an insurance agent, and the $100 is called the premium. A typical insurance agent has no idea what the probability is for the event that is being insured against, even though the customer needs that piece of information in order to decide whether to buy the insurance. And even if the agent knew that information, a typical customer wouldn't know what to do with it. But the insurance company does know the probability, and the fact that insurance companies make a profit is the evidence that the probabilities favor them, not their customers.

The agent tells the customer “We will protect your house against fire.”; that's a lie; the insurance company cannot prevent fire. Or maybe the agent says “We will protect you against loss due to fire.”; that's a bit better, but still false; the insurance company cannot prevent the loss of your house or belongings. All they can do is pay off their bet.

With this story, I have tried to show that fire insurance is a bet you make with an insurance company. Similarly all other insurance is a bet. To decide whether to bet (buy insurance), you need to know the buy-in (premium), payoff (amount of insurance), the probability of winning (gaining the payoff), and the amount of your available funds. Whether the event is unpleasant (house burns down), pleasant (win Field's medal), or neutral (the falling balls showed my sequence of digits) is irrelevant for making a rational, mathematically sound decision. But there is a psychological reason why unpleasant events are offered as payoff triggers: it's nice to balance a bad day (house burns down) with a good day (winning a large bet). Apparently this psychological fact outweighs mathematics, especially for that large part of the population that can't do math.

Some will rebut: It's not just “psychological”, and it's not just “nice”; I can't afford to lose my house without also winning a compensating bet. That's a good point, but it doesn't justify making a bad bet. Not all insurance is a bad deal. Not-for-profit insurance, sometimes run by government, sometimes run as a co-operative, can be a fair deal. Self-insurance, otherwise known as saving for a rainy day, is a fair deal. But buying insurance from a profit-making insurance company is a bad deal, and the more profitable the company, the worse the deal.

There is a portion of the population that is mostly rational, maybe even knows enough math, but cannot break free from the constant psychological bombardment from insurance companies. The companies say “Don't take a chance; buy insurance.” when actually buying insurance is gambling. They invoke the fear and sadness of the unpleasant event to keep you from making a rational decision. And they invoke some questionable mathematics. An example is “nonlinear utility”: your first dollar is worth more to you than your 500,000th dollar. Your first dollars keep you alive, and your last dollars just add marginally to your comfort. You need to protect your ability to live (by keeping a roof over your head), and therefore, they conclude, it is worth buying insurance even if the probability of “winning” the payoff doesn't justify paying the premium from a purely gambling point of view. But this argument makes no sense. If you have very little money, then you can't afford to be giving it to an insurance company. If you have lots of money, you don't need the insurance. Nothing justifies making a bad bet.

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Addendum 2020-1-6

When we are deciding whether to buy insurance, should we take into account the fact that the money paid as premium could instead be invested for compound interest? Are interest rates relevant to the decision whether to buy insurance? Should we consider just the current interest rate? Or the interest rate over the time for which the premium pays for insurance? Or the interest rates from now on forever, since the money, if not paid today as premium, will earn interest forever? And if we are going to consider the interest rate, should we also consider the inflation rate, since our interest-increased money will have inflation-decreased buying power?

Think about a simple bet: Alice and Bob each put $1 into a pot (that's the buy-in), then they flip a coin, and the winner takes the pot ($2 payout). The coin has a one-half probability for each of its two possible outcomes, so the bet is fair. But wait: should Alice check on the current and future interest rates, because if she doesn't buy in she can invest that dollar? And if she wins, she can invest $2. And of course the same question applies to Bob. The answer for this simple bet is no, you don't need to consider interest rates. What you do with your $1 if you don't buy in, and what you do with your $2 if you buy in and win, is not relevant to the fairness of the bet.

Now think about the same simple bet, but with a delay in the payoff. Alice and Bob each put $1 in the pot now, and a coin is flipped sometime between now and a year from now, and the payoff to the winner is a year from now. If we consider only the buy-in, the probability of winning, and the payoff, and we ignore interest rates, it's a fair bet. Now let's consider an interest rate of 5%. If Alice and Bob don't bet, they can each invest, and at the end of a year, they each have \$1.05. If they do bet, the pot can be invested for a year, and the payoff to the winner is \$2.10. So, Alice reasons, at the end of a year, my buy-in money is worth \$1.05, my probability of winning is 50%, and my payoff if I win is \$2.10. It's fair. And Bob says the same. The point is that interest rates are the same for everyone, so it doesn't matter whether the calculation is made before or after interest has been earned.

Suppose you buy a year of life insurance from an insurance company. The premium is paid now, the bet is decided sometime during the next year if you die, and at the end of the year if you live. The payout is made sometime after the bet is decided. If you don't buy the insurance, you can invest what you would have paid as premium. If you do buy the insurance, the insurance company most certainly invests your premium. If you die, what your estate gets paid depends on the terms of the agreement. Suppose the amount is indexed to inflation, and suppose inflation is equal to the interest rate; then the situation is the same as the bet in the previous paragraph: it doesn't matter whether the calculation is made before or after interest and inflation have occurred. When you are deciding whether to buy the insurance, you don't know what the interest and inflation rates will be, so you must make the calculation based on current amounts, ignoring interest and inflation.

Usually the amount your estate gets is fixed, not indexed to inflation. In that case, the insurance company calculates its expected gain due to interest, and sets your premium with that expected gain in mind. If it is a fair, not-for-profit company, it sets your premium so that the premiums plus interest equal the payoffs plus administrative costs. But you don't need to know all those amounts. You just need to know the amount of your premium, the probability of dying, the amount the payoff would be if it were paid now, and your available funds.

Fire insurance is more complicated because the payout depends on the amount of your loss, but all the same principles apply. For all insurance, you know that if the company makes a profit, insurance is not a fair bet; if the company (or government insurance) is not-for-profit, then insurance is fair.

Addendum 2020-1-9

In an email to me, Josh Jordan has pointed out that I have used the terms “good bet”, “fair bet”, and “bad bet” without defining them. Define “good bet” as a bet where the amount you stand to gain (payoff minus buy-in) times the probability of winning is greater than the amount you stand to lose (buy-in) times the probability of losing. Define “fair bet” as a bet where the amount you stand to gain times the probability of winning is equal to the amount you stand to lose times the probability of losing. Define “bad bet” as a bet where the amount you stand to gain times the probability of winning is less than the amount you stand to lose times the probability of losing. Despite the use of the words “good”, “fair”, and “bad”, these definitions do not determine how much you should bet if you have a choice; the amount you choose to bet may depend on your available funds. If you are offered only a single betting amount, then your decision whether to bet that amount may depend on your available funds.

I am also grateful to Josh Jordan for bringing the Kelly Criterion to my attention. In the case of a good bet, the Kelly Criterion tells you what fraction of your available funds to bet in order to maximize your winnings in the long run (betting repeatedly). In the case of a bad bet, if you bet any positive amount, you will ultimately go bankrupt.

Josh Jordan raises a similar question about one-time-only bets. His example, in his own words, is as follows.

> “Suppose you are offered the following bet on a biased coin that comes up heads 60% of the time: if it comes up heads, you win \$100,000; otherwise, you lose \$100,000. That is a good bet by your definition. Now, two questions:
> (1) Your net worth is \$100,000. Should you take the bet?

(2) Your net worth is \$10 billion. Should you take the bet?

This example is meant to illustrate that, when deciding whether to bet, there's more relevant financial information to consider than merely the buy-in, payoff, and probability of winning."

Let me change the problem slightly (but I will put it back to Jordan's version in a moment).

Suppose you are offered the following bet on a biased coin that comes up head with probability p : if it comes up head, you win $\$x$, otherwise, you lose $\$x$, for your choice of x . Now, two questions:

(3) Do you want to take the bet?

(4) If so, what is your choice of x ?

We can eliminate question (3) because one possible answer to question (4) is \$0, which is equivalent to answering "no" to question (3). The answer to question (4) depends on p . If $p<50\%$ then it's a "bad bet", so I choose $x=0$. If $p=50\%$ then it's a "fair bet", and if $p>50\%$ then it's a "good bet", but how much should I bet? In the spirit of the Kelly Criterion (even though I am not maximizing my winnings in the long run), I might decide that the amount of my bet depends on p and my net worth w as follows: $x = (2\times p - 100\%)\times w$. For example, if $p=60\%$, I will bet 20% of my net worth. If $p=90\%$, I will bet 80% of my net worth. Someone else may reasonably choose a different formula, but any formula must have the property that as p increases from 50% to 100%, the fraction of net worth that is bet increases from 0% to 100%.

To return to Josh Jordan's problem, in his question (1), $p=60\%$ and $w=\$100{,}000$. I am willing to bet $x = (2\times p - 100\%)\times w = (2\times 60\% - 100\%)\times\$100{,}000 = \$20{,}000$, not $\$100{,}000$, which is the bet that I am offered. So my answer to question (1) is no. In question (2), $p=60\%$ and $w=\$10\text{bn}$. I am willing to bet $x = (2\times p - 100\%)\times w = (2\times 60\% - 100\%)\times\$10\text{bn} = \$2\text{bn}$, which is more than $\$100{,}000$, which is the bet that I am offered. So my answer to question (2) is yes.

In Josh Jordan's problem, the amount you might lose equals the amount you might win. In real problems, the amount you lose may differ greatly from the amount you stand to win. An insurance premium costs much less than the payoff. A lottery ticket costs much less than you might win.

Let's look at lotteries first. The probability of winning is p . You have amount w available for buying tickets. The ticket price is x . The payoff if you win is y . How many tickets should you buy? Let's call the number of tickets you should buy n . The input to our calculation is p , w , x , and y , and the output is n . My formula is

$$n = (p\times y - x)/(y-x)\times w/x$$

It says that when $p=100\%$, which means you are sure to win on every ticket, you should buy all the tickets you can afford: $n = w/x$. It says that when $y = 2\times x$, which means you have Josh Jordan's problem, you should buy $(2\times p - 100\%)\times w/x$ tickets. In a real lottery, the probability p of winning is very small $p < x/y$, and the formula says you should buy a negative number of tickets; you should be selling tickets, not buying them.

Now let's look at insurance. You have amount w as available funds. The probability that the event will happen and trigger the payoff is p . You decide what payoff (how much insurance) you want; call that y . The insurance company tells you what the buy-in (premium) costs; call that x . If $(p\times y - x)/(y-x)\times w/x$ is greater than or equal to 1 , the insurance is worthwhile, so buy it. If $(p\times y - x)/(y-x)\times w/x$ is less than 1 , the insurance is not worthwhile, so don't buy it. If $(p\times y - x)/(y-x)\times w/x$ is between 0 and 1 , perhaps a smaller amount of insurance might be worthwhile. For realistic values of p , y , and x , $p < x/y$ and so $(p\times y - x)/(y-x)\times w/x$ is negative; you should be selling insurance, not buying it.