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a Note on an Equation due to Euler

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The equation in question is

(0) $\quad ... + a\times b^3 + a\times b^2 + a\times b^1 + a\times b^0 + a\times b^{-1} + a\times b^{-2} + a\times b^{-3} + ... \ = \ 0$

for any $b\neq 1$. Let this doubly infinite sum be x . Multiplying each term in the sum by b gives us exactly the same sum, so

(1) $\quad x\times b = x$

And since $b\neq 1$, therefore $x=0$.

I'll look at the special case $a=1$ and $b=2$, and I'll break the sum into two halves. Let y be the right half.

(2) $\quad y \ = \ 2^{-1} + 2^{-2} + 2^{-3} + ...$

If we multiply every term in that sum by 2 , we obtain exactly same sum but with an extra term. So

(3) $\quad y\times 2 \ = \ 2^0 + y$

which we solve to find $y=1$. This is a well-known and well-agreed result.

Let z be the left half.

(4) $\quad z \ = \ 2^0 + 2^1 + 2^2 + 2^3 + ...$

If we multiply every term in that sum by 2 , we obtain exactly same sum but with the first term missing. So

(5) $\quad 2^0 + z\times 2 \ = \ z$

which we solve to find $z = -1$. Even though the reasoning to obtain (5) is identical to the reasoning to obtain (3), this result is not well-known and not well-agreed. But it is an instance of Euler's equation (0):

(6) $\quad x \ = \ z+y \ = \ (-1)+(1) \ = \ 0$

And there is an interesting kind of rational arithmetic called quote notation based on it.

Actually, (1) has 3 solutions: $x=\infty$, $x=0$, $x=-\infty$. And (3) has 3 solutions: $y=\infty$, $y=1$, $y=-\infty$. And (5) has 3 solutions: $z=\infty$, $z=-1$, $z=-\infty$. We are free to choose any solution without inconsistency. The reason that mathematicians may prefer the solutions $y=1$ and $z=\infty$ is called continuity. If we look at a finite sum of n terms, and then take the limit as n increases toward ∞ (denoted $\Updownarrow n\cdot$), we find

$$(\Updownarrow n\cdot \ 2^{-1} + 2^{-2} + 2^{-3} + ... + 2^{-n}) \ = \ 1$$

$$(\Updownarrow n\cdot \ 2^0 + 2^1 + 2^2 + 2^3 + ... + 2^{n-1}) \ = \ \infty$$

These limits do not compel us to say $y=1$ and $z=\infty$, but they provide a reason for mathematicians to prefer those solutions.

We could test whether $y=1$ or $y=\infty$ if we could build a mechanism that iterates (loops), with each iteration taking half as long as the previous iteration. We then run the mechanism and see if it finishes after 1 unit of time or keeps going. We cannot build such a mechanism, but nature has built such a mechanism; it is called a black hole. To an outside observer watching an object fall into a black hole, the object takes forever to reach the event horizon (a specific distance from the center of the black hole), and it never gets past that horizon. But to someone who is falling into a black hole, they reach the event horizon in finite time and continue falling inward. So which is it: finite or infinite time to reach the event horizon? The answer depends upon your observation point. Are you watching from outside, or are you the object falling in?

(I report this surprising result of current physics, but I find it impossible to accept. See Time Dilation.)