### Twisted Self-Reference

There is a standard argument, appearing in many textbooks, in a variety of different notations, that is supposed to prove that the Halting Problem is incomputable. It considers a procedure, let's call it \( \text{diag} \), whose only action is

\[
\text{if } \text{halts}(\text{diag}) \text{ then } \text{infiniteloop} \text{ else } \text{terminate} \text{ fi}
\]

where \( \text{halts} \) is a function that determines whether execution of a program terminates, \( \text{infiniteloop} \) is an infinite loop, and \( \text{terminate} \) terminates. If \( \text{halts} \) says that execution of \( \text{diag} \) is terminating, then it's nonterminating; and if \( \text{halts} \) says that execution of \( \text{diag} \) is nonterminating, then it's terminating. Whatever \( \text{halts} \) reports for \( \text{diag} \), it is wrong; there cannot be a halting program. I will call this argument the “twisted self-reference” proof. In the paper *Epimenides, Gödel, Turing: an Eternal Gölden Tangle*, I argue that the twisted self-reference proof does not prove that halting is incomputable; rather it proves that the specification “Write a program in language L that determines whether execution of any program in language L terminates.” is inconsistent, or self-contradictory.

### Diagonalize Then Reduce

There is another argument, which I will call “diagonalize-then-reduce”, that is supposed to prove that the Halting Problem is incomputable without using any self-reference. Here is a version of it.

Choose a programming language. All programs in that language are finite sequences of characters, although not all finite sequences of characters are programs. Execution of a program may read a sequence of characters as input, and may write a sequence of characters as output. Reading does not have to precede writing; they can be mixed. The input sequence may be empty, or a finite number of characters, or an infinite number of characters. Likewise the output sequence. Execution may terminate, or it may run forever.

Let \( C \) be a finite character set, and let \( C^* \) be the set of all finite sequences of characters. Define the mathematical function \( D \) (not a program) called “diagonal” as follows.

\[
D : C^* \to \{\text{“red”}, \text{“blue”}\}
\]

\[
D(p) = \text{“red”} \text{ if } p \text{ is a program and execution of } p \text{ on input } p \text{ writes “blue” and then terminates}
\]

\[
\text{“blue” otherwise}
\]

\( D(p) = \text{“red”} \) when

- \( p \) is a program, and execution of \( p \) writes “blue” and terminates, with or without reading the entire input \( p \)

\( D(p) = \text{“blue”} \) when

- \( p \) is a program, and execution of \( p \) writes nothing and terminates, with or without reading the entire input \( p \)
- \( p \) is a program, and execution of \( p \) writes anything other than “blue” and terminates, with or without reading the entire input \( p \)
- \( p \) is a program, and execution of \( p \) reads the entire input \( p \) and waits forever for more input, regardless of what is written
- \( p \) is a program, and execution of \( p \) on input \( p \) does not terminate, regardless of what is read or written
- \( p \) is a not a program

Let \( \text{prog} \) be a program. Does \( \text{prog} \) implement \( D \)? Implementation means:

- For all \( p \) in \( C^* \), if \( D(p) = \text{“red”} \) then execution of \( \text{prog} \) on input \( p \) writes “red” and terminates.
- For all \( p \) in \( C^* \), if \( D(p) = \text{“blue”} \) then execution of \( \text{prog} \) on input \( p \) writes “blue” and terminates.

However, if execution of \( \text{prog} \) on input \( \text{prog} \) writes “red” and terminates, then \( D(\text{prog}) = \text{“blue”} \), not “red”. And if execution of \( \text{prog} \) on input \( \text{prog} \) writes “blue” and terminates, then \( D(\text{prog}) = \text{“red”} \), not “blue”. So \( \text{prog} \) does not implement \( D \). Since \( \text{prog} \) was an arbitrary program, \( D \) is incomputable.
Define the mathematical function $H$ (not a program) called “halting” as follows.

$$H: \mathbb{C}^* \to \{\text{"yes"}, \text{"no"}\}$$

$H(p) = \text{"yes"}$ if $p$ is a program and execution of $p$ on input $p$ terminates

“no” otherwise

This halting function reports the halting status for each program $p$ on only a single input $p$. $H(p) = \text{"yes"}$ includes the possibility that $p$ is a program and execution of $p$ does not read the entire input $p$. $H(p) = \text{"no"}$ includes the possibility that $p$ is a program and execution of $p$ reads the entire input $p$ and waits forever for more input.

Assume (for contradiction) that $H$ is computable. Then $H$ is implemented by some program $\text{halts}$. If the programming language is sufficiently expressive (Turing-Machine equivalent), as every general-purpose programming language is, we can compute $D(p)$ as follows.

Read the input and save it as $p$. Execute $\text{halts}$ on input $p$, but don't output. If the output from executing $\text{halts}$ on $p$ would be “no”, output “blue”. If the output from executing $\text{halts}$ on $p$ would be “yes”, execute program $p$ on input $p$, but don't output. If the output from executing $p$ on $p$ would be “blue”, output “red”. If the output from executing $p$ on $p$ would be anything other than “blue”, output “blue”.

We thus compute $D$. But $D$ is incomputable. Therefore $H$ is incomputable.

**Discussion**

We began by choosing a programming language; call it L. Mathematical function $D$ is defined by diagonalizing over the programs of language L. The definition of mathematical function $D$ is not self-referential, and it is consistent. We then ask whether $D$ is implemented by a program in L; let's call it $\text{prog}$. Program $\text{prog}$ must implement $D$, which is defined over programs in L, including $\text{prog}$, with a twist so that $D$ differs from $\text{prog}$. Program $\text{prog}$ is defined with a twisted self-reference; its specification is inconsistent; there is no such program.

But we cannot conclude that $D$ is incomputable, because we have not asked whether $D$ can be implemented in a programming language other than the one over which $D$ is defined.

Consider the question “Can an L program correctly answer “no” to this question?”. It is easy to write an L program whose execution prints “yes”, but that answer says that “no” is the correct answer. There is another L program that prints “no”, but that answer says that no L program can do what it is doing (printing “no” in answer to the question). There is no program in language L that answers the question correctly. But there is a program in language M that answers that same question correctly: it prints “no”, saying that no L program can correctly answer the question. Due to the twisted self-reference, the task is impossible for an L program. But it is not incomputable; it can be answered by an M program. Symmetrically, the question “Can an M program correctly answer “no” to this question?” cannot be correctly answered by an M program, but it can be correctly answered by an L program.

Likewise function $D$ cannot be computed by an L program due to the twisted self-reference. But that does not prevent $D$ from being computed by an M program. The conclusion that $D$ is incomputable is unwarranted.

We have done the diagonalization; now comes the reduction. Mathematical function $H$ is defined as the halting function for programs in language L. Its definition is not self-referential, and it is consistent. The final paragraph says: if we could compute halting, then we could compute $D$. But we can't compute $D$. So we can't compute halting. halting is incomputable. To be more precise, the final paragraph means: if we could write an L program to compute halting for all L programs, then we could write an L program to compute $D$. But we can't write an L program to compute $D$. So we can't write an L program to compute halting for all L programs. We cannot conclude that halting is incomputable. We can conclude only that the specification “Write an L program to compute halting for all L programs.” is inconsistent. That conclusion does not prevent halting for language L from being computed by a program in a language other than L.

other papers on halting
Appendix in reply to a challenge, added 2016-11-13

My “Discussion” section contains the statement “But we cannot conclude that \( D \) is incomputable, because we have not asked whether \( D \) can be implemented in a programming language other than the one over which \( D \) is defined.”. A friend suggested the following argument, concluding that \( D \) cannot be implemented in any programming language.

Define mathematical function \( D \) as follows: for all programs \( p \) in language \( L \), \( D(p) \neq p(p) \). Function \( D \) differs from all programs in \( L \) on at least one input. Therefore \( D \) is not computed by any program in \( L \). Let \( C \) be a program in language \( M \) that computes \( D : \) for all programs \( p \) in \( L \), \( C(p) = D(p) \). Then there is an equivalent program \( B \) in \( L \): for all programs \( p \) in \( L \), \( B(p) = C(p) \). Now calculate:

\[
C(B) = D(B) \quad \text{use definition of } C \\
\neq B(B) \quad \text{use definition of } D \\
= C(B) \quad \text{use definition of } B
\]

Hence \( C(B) \neq C(B) \), which is a self-contradiction. Conclusion: there is no program in \( M \) that computes \( D \).

There are some minor problems with this argument. To pass a program as data to a function or to another program, you need to encode it (as a number or character string). That problem is trivial to fix, and I'll ignore it. Another problem is that if execution of program \( p \) does not terminate on input \( p \) , then \( p(p) \) is undefined. That problem may seem to be fixed by saying that \( D(p) \) can be any result for that case, although there are problems with that fix; but I'll ignore that problem too. Another problem is that \( D(p) \neq p(p) \) does not say what the value of \( D(p) \) is; only what it isn't. That problem is fixed by choosing a specific result for \( D(p) \) except when \( p(p) \) is also that result, and for that case choosing one other result. Equivalently, we restrict programs to those with a binary result, and define \( D \) to have a binary result. So I'll ignore that problem too.

When we arrive at the contradiction \( C(B) \neq C(B) \), we are compelled to withdraw some assumption we made leading to the contradiction. The assumption chosen is: “\( C \) is a program in \( M \) that computes \( D \)”. But there is another candidate. The statement “there is an equivalent program \( B \) in \( L \)” contains a hidden assumption that I think is wrong. I'll explain in a moment.

Here's the same argument as above, but I simplify by getting rid of the function's parameter, making it a constant.

Define mathematical constant \( D \) as the correct answer to the question “Can an \( L \) program correctly answer “no” to this question?”. If an \( L \) program can correctly answer “no”, then \( D=\text{“yes”} \). If an \( L \) program cannot correctly answer “no”, leaving “yes” as the correct answer, then \( D=\text{“no”} \). Constant \( D \) is defined such that if an \( L \) program says \( B \), then \( B \) is not the correct answer: \( D\neq B \). Assume there is a program in \( M \) that gives the correct answer \( C \); then \( C=D \). Then there is an equivalent program \( B \) in \( L \) that gives the same answer: \( B=C \). Now calculate:

\[
C = D \quad \text{use definition of } C \\
\neq B \quad \text{use definition of } D \\
= C
\]

Hence \( C\neq C \), which is a self-contradiction. Conclusion: there is no program in \( M \) that correctly answers \( D \).

The conclusion is wrong: there is a program in \( M \) that answers correctly: it prints “no”. Where does the argument go wrong? The argument says “there is an equivalent program \( B \) in \( L \) that gives the same answer: \( B=C \)”. Indeed there is a program in \( L \) that prints the same answer “no”, but when a program in \( L \) prints “no”, it's incorrect.

Likewise in the previous argument where \( D \) is a function with a parameter. If there is a program \( C \) in \( M \) that computes \( D \), then yes, there is an “equivalent” program in \( L \) which, for each input, gives the same output. But that \( L \) program doesn't compute \( D \).

I put the word “equivalent” in quotation marks because I think it is ambiguous. It might mean “for each input gives the same output”; let's call that extensional equivalence. Or it might mean “satisfies the same specification”; let's
call that “intensional equivalence”. Most of the time, intensional and extensional equivalence are the same thing. They may differ when there's a self-reference. The above proofs pivot on the word “equivalence”.

In the simplified version where \( D \) is a constant, the calculation \( C = D \neq B = C \) uses an intensional step: \( D \neq B \). \( D \) is defined to differ from \( B \). A reasonable person might say: first show me \( B \), then we can define \( D \) to be the other answer. That would be an extensional definition. But we cannot show \( B \) because both answers are incorrect when said by an L program. So \( D \) is not defined extensionally. It is defined intensionally as differing from \( B \), whatever \( B \) is.

Likewise in the version where \( D \) is a function with a parameter. The calculation \( C(B) = D(B) \neq B(B) = C(B) \) uses an intensional step: \( D(B) \neq B(B) \). \( D(p) \) is defined to differ from \( p(p) \), and so \( D(B) \neq B(B) \). A reasonable person might say: first show me \( B(B) \), then we can define \( D(B) \) to be the other answer. That would be an extensional definition. But we cannot show \( B(B) \). So \( D(B) \) is not defined extensionally. It is defined intensionally as differing from \( B(B) \), whatever \( B(B) \) is.

When we come to the self-contradiction, the assumption that I would flag as being wrong is the hidden assumption that intensional definitions are equivalent to extensional definitions. Normally they are equivalent, but in the presence of a self-reference, they may not be equivalent, and in this case, they are not equivalent.