Three Variations on a Theme by Collatz

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Abstract

The Collatz problem is reformulated in three ways as a non-arithmetic problem in first-order logic.

1 Introduction

Among the many unsolved problems in mathematics, the Collatz conjecture holds a position of honour. It is a puzzle that looks quite simple, even trivial, on the surface, and yet somehow manages to lie beyond the reach of all the mathematicians who have tried to solve it. (See [4, 5] for surveys of work on it.) It can be stated in terms of the following program:

```
while n > 1 do
  if n is even
    then n := n/2
    else n := 3*n + 1
  end if
end while
```

The question is this: will this program terminate regardless of the initial value of the integer variable `n`? The conjecture is that it will, and this has been confirmed so far for all values of `n` up to $17 \times 2^{58} > 4 \times 10^{18}$. But despite the best efforts of many mathematicians, there is still no proof that it will always do so. Paul Erdös famously quipped: "Mathematics is not yet ready for such puzzles." What makes the puzzle so baffling is how little mathematics appears to be involved: just the most basic integer arithmetic, not even prime numbers or exponentiation!

In this technical note, we try to put the arithmetic aside and present three purely logical reformulations of the puzzle. This may not contribute much towards actually solving the puzzle, but the goal here is somewhat more esthetic: the logical accounts allow us to get close and to admire it from a slightly different angle.

To simplify matters in what follows, we first observe that if n is odd ($n = 2i+1$), then $3n+1$ must be even ($6i+4$), and so the next pass through the loop in the above program will always divide it by 2 (resulting in $3i + 2$). So we can skip a pass through the loop and rewrite the program as follows:

```
while n > 1 do
  if n is odd then n := 3*n + 1
  n := n/2
end while
```

We can of course make the Collatz conjecture precise in a way that does not depend on programs:

Definition 1 *Define the function* $C \in [\mathbb{N} \to \mathbb{N}]$ *by*

$$C(n) = \begin{cases} i & \text{if } n = 2i \\ 3i + 2 & \text{if } n = 2i + 1 \end{cases}$$

Conjecture (Collatz) *For every* $n > 0$*, there is a* $k \geq 0$*, such that* $C^k(n) = 1$.

For example, for $n = 5$, the k is 4, since $C(5) = 8$, $C(8) = 4$, $C(4) = 2$, $C(2) = 1$.

2 Theme

Obviously, it is possible to formulate the Collatz conjecture in a logical language when that language already includes arithmetic. Let us call a logical interpretation of a language $\mathcal{L}$ *arithmetic* if the domain is $\mathbb{N}$, and the following symbols (assuming they are in $\mathcal{L}$) get their usual interpretations: the constant symbols 0 and 1, the binary function symbols $+$ and $\times$, and the binary predicate symbol $=$. Then we say that a sentence of $\mathcal{L}$ is *arithmetically valid* if it is true in all arithmetic interpretations. Now consider the following theory:

Definition 2 *Let* CA_0 *be the conjunction of the following first-order sentences:*

1. $R(1)$
2. $\forall x.\, R(x) \supset R(x + x)$
3. $\forall x.\, R(x + x + x + 1 + 1) \supset R(x + x + 1)$

(The + function is written here in infix form and parentheses have been suppressed in the usual way.) It is not hard to see that the Collatz conjecture is true iff the sentence $(CA_0 \supset \forall x. R(x+1))$ is arithmetically valid.

This is perhaps not too surprising since this form of logic includes arithmetic. For example, if the language has just the arithmetic symbols 0, 1, +, × and =, a sentence is arithmetically valid iff it is true in the standard interpretation. So we are in the domain of Peano arithmetic and the set of arithmetically valid sentences is not even recursively enumerable! (See [3], for example.)

If we drop the × symbol (and note that it was not needed in CA_0 above), then we are in the domain of Presburger arithmetic, where the set of arithmetically valid sentences is recursive [3]. However, if we include even a single unary predicate (as we did with the R predicate above), the resulting arithmetically valid sentences need not be recursive in general. (See [1] for results on weak arithmetics.)

What this shows is that there is a strong (and well known) connection between arithmetic and certain forms of logic. This is seen most clearly in the case of second-order logic where arithmetic can be defined in the language. For example, starting with just the constant symbol 0 and a unary function symbol s, we can restrict the domain to be isomorphic to the natural numbers by considering logical entailments of the following sentence (the so-called "induction axiom"):

$$\forall N. \, [N(0) \land \forall x. \, N(x) \supset N(s(x))] \; \supset \; \forall x. \, N(x).$$

We can define addition by letting $\alpha + \beta = \gamma$ stand for the following formula:

$$\forall P. \, [\forall x \; P(0, x, x) \; \land \; \forall x, y, z. \; P(x, y, z) \supset P(s(x), y, s(z))] \; \supset \; P(\alpha, \beta, \gamma)$$

Multiplication can defined similarly (using addition). It follows that we are able to write a sentence of second-order logic that is logically valid iff the Collatz conjecture is true. So we do not really need the notion of arithmetic validity; ordinary validity in second-order logic will do the job. (It follows, of course, that the logically valid sentences of second-order logic are also not recursively enumerable.)

But returning to the Collatz conjecture, the question is not whether it can be stated in a language that can already express all of arithmetic; the question is how to reformulate it in less expressive formalisms, which we now turn to.

3 First Variation

In this first variation, we consider the following logical language:

Definition 3 *By $\mathcal{L}_1$ we mean the logical language with one constant symbol o, one unary function symbol s, one unary predicate symbol R, and one ternary predicate symbol T.*

What we will be concerned with are the logically valid sentences of $\mathcal{L}_1$ and, in particular, the logical consequences in $\mathcal{L}_1$ of the following:

Definition 4 *Let CA_1 be the conjunction of the following four first-order sentences of $\mathcal{L}_1$ (where variables x, y, z are universally quantified from the outside):*

1. $T(o, o, o)$
2. $T(x, y, z) \supset T(s(x), s(s(y)), s(s(s(z))))$
3. $R(s(o))$
4. $T(x, y, z) \;\supset.\; (R(x) \supset R(y)) \wedge (R(s(s(z))) \supset R(s(y)))$

We can see what is intended by CA_1 by looking at the following arithmetic interpretation of $\mathcal{L}_1$:

Definition 5 *Let $\mathfrak{G}_1$ be the interpretation of $\mathcal{L}_1$ defined as follows:*

- *the domain of* $\mathfrak{G}_1$ *is* $\mathbb{N}$*;*
- $o^{\mathfrak{G}_1} = 0$*;*
- $s^{\mathfrak{G}_1} = \{(n, n+1) \mid n \geq 0\}$*;*
- $T^{\mathfrak{G}_1} = \{(n, 2n, 3n) \mid n \geq 0\}$*;*
- $R^{\mathfrak{G}_1} = \{n \mid n > 0$ *and for some* $k \geq 0,\ C^k(n) = 1\}$.

Clearly $\mathfrak{G}_1 \models CA_1$ and the Collatz conjecture is true iff $\mathfrak{G}_1 \models \forall x.\ R(s(x))$. But unlike before, we are now interested in ordinary logical validity in $\mathcal{L}_1$, which means allowing for interpretations of $\mathcal{L}_1$ that are less standard, like this one:

Definition 6 *Let $\mathfrak{B}$ be the interpretation of $\mathcal{L}_1$ defined as follows:*

- *the domain of* $\mathfrak{B}$ *is* $\mathbb{N} \times \mathbb{N}$*;*
- $o^{\mathfrak{B}} = (0, 0)$*;*
- $s^{\mathfrak{B}} = \{((n, m), (n, m+1)) \mid n, m \geq 0\}$
- $T^{\mathfrak{B}} = \{((n, m), (2m, 2m), (3n, 3m)) \mid n, m \geq 0\}$
- $R^{\mathfrak{B}} = \{(2^n, 2^n) \mid n \geq 0\} \cup \{(0, n) \mid n > 0\}$.

In this case, the domain elements are pairs (n, m). We can think of the $(0, m)$ pairs as corresponding to ordinary numbers, and satisfying the intended properties with respect to s, and T, as in $\mathfrak{G}_1$. But pairs $(1, m)$ are like non-standard integers: they have successors and can be doubled and tripled, but are missing certain properties. For example, pairs of the form $(1, m)$ are not "even" in that there is no u and v such that $(u, (1, m), v) \in T^{\mathfrak{B}}$, and not "odd" in that there is no u and v such that $(u, (1, m+1), v) \in T^{\mathfrak{B}}$. However, we do have the following:

Lemma 1 $\mathfrak{B} \models CA_1$.

Proof: The first three sentences of CA_1 are obviously satisfied by $\mathfrak{B}$.

For $CA_1^{(4)}$, note that if $x \in R^{\mathfrak{B}}$, then $x = (0, n)$ or $x = (2^k, 2^k)$. So if $(x, y, z) \in T^{\mathfrak{B}}$, then $y = (0, 2n)$ or $y = (2^{k+1}, 2^{k+1})$, and $z = (0, 3n)$ or $z = (3 \cdot 2^k, 3 \cdot 2^{k+1})$. Either way, $y \in R^{\mathfrak{B}}$. Furthermore, if $\sigma(\sigma(z)) \in R^{\mathfrak{B}}$, where $\sigma = s^{\mathfrak{B}}$, then $z = (0, 3n)$ since $(3 \cdot 2^k, 3 \cdot 2^k + 2) \notin R^{\mathfrak{B}}$. But in this case $y = (0, 2n)$, and so $(0, 2n + 1) \in R^{\mathfrak{B}}$. ∎

As a consequence, we have the following:

Theorem 1 *The sentence* $(CA_1 \supset \forall x.\, R(s(x)) \supset R(s(s(x))))$ *is not logically valid.*

Proof: By Lemma 1, $\mathfrak{B} \models CA_1$. Now consider $x = (1, 0)$. We have $(1, 1) \in R^{\mathfrak{B}}$ but $(1, 2) \notin R^{\mathfrak{B}}$. So $\mathfrak{B} \not\models \forall x.\, R(s(x)) \supset R(s(s(x)))$. ∎

Corollary 1 *The sentence* $(CA_1 \supset \forall x.\, R(s(x)))$ *is not logically valid.*

So the existence of "non-standard" models like $\mathfrak{B}$ shows that CA_1 is not strong enough to force $R(s(x))$ to be true for every x. In fact, it is not even strong enough to force an inductive version of this: just because $R(s(x))$ holds, it need not be the case that its immediate successor $R(s(s(x)))$ holds.

It might then be argued that CA_1 is just too weak, that it fails to capture the Collatz puzzle. But it does capture it, as the following theorem shows.

Definition 7 *For every $n \geq 0$, let $\underline{n}$ stand for the ground term of $\mathcal{L}_1$ consisting of n applications of s to o, that is, the term $s(s(s(\cdots o \cdots)))$, with s repeated n times.*

Lemma 2 *Let $\mathfrak{M}$ be an interpretation such that $\mathfrak{M} \models CA_1$. Then for any $n \geq 0$, $\mathfrak{M} \models T(\underline{n}, \underline{2n}, \underline{3n})$.*

Proof: By induction on n.

Base case: If $n = 0$, then $\underline{n} = \underline{2n} = \underline{3n} = o$. So by $CA_1^{(1)}$, $\mathfrak{M} \models T(\underline{n}, \underline{2n}, \underline{3n})$.

Induction step: Suppose that $n \geq 0$ and that $\mathfrak{M} \models T(\underline{n}, \underline{2n}, \underline{3n})$. Then, by $CA_1^{(2)}$, $\mathfrak{M} \models T(s(\underline{n}), s(s(\underline{2n})), s(s(s(\underline{3n}))))$. Therefore, $\mathfrak{M} \models T(\underline{n+1}, \underline{2n+2}, \underline{3n+3})$. ∎

Lemma 3 *Let $\mathfrak{M}$ be an interpretation such that $\mathfrak{M} \models CA_1$. Then for every $k \geq 0$ and $n > 0$, if $C^k(n) = 1$ then $\mathfrak{M} \models R(\underline{n})$.*

Proof: By induction on k.

Base case: If $k = 0$ and $C^k(n) = 1$ then $n = 1$ and $\mathfrak{M} \models R(\underline{n})$ by $CA_1^{(3)}$.

Induction step: Suppose $k \geq 0$ and for every $n > 0$, if $C^k(n) = 1$ then $\mathfrak{M} \models R(\underline{n})$. Now suppose $n > 0$ and $C^{k+1}(n) = 1$. This means there is a $u > 0$ such that $C(n) = u$ and $C^k(u) = 1$. By the induction hypothesis, $\mathfrak{M} \models R(\underline{u})$. There are two cases for n:

- n is even, and $n = 2u$. Then $\mathfrak{M} \models T(\underline{u}, \underline{2u}, \underline{3u})$ by Lemma 2. Since $\mathfrak{M} \models R(\underline{u})$, it follows that $\mathfrak{M} \models R(\underline{2u})$ by $CA_1^{(4)}$.
- n is odd, say $n = 2v + 1$, where $u = 3v + 2$. Then $\mathfrak{M} \models T(\underline{v}, \underline{2v}, \underline{3v})$ by Lemma 2 Since, $u = 3v + 2$, $\mathfrak{M} \models R(s(s(\underline{3v})))$. So $\mathfrak{M} \models R(s(\underline{2v}))$ by $CA_1^{(4)}$.

This completes the proof. ∎

Now we can state the main result of this section, showing that ordinary logical validity is sufficient to capture the Collatz conjecture:

Theorem 2 *Collatz is true iff for every $n > 0$, the formula $(CA_1 \supset R(\underline{n}))$ is a valid sentence of first-order logic.*

Proof: In the if direction, suppose that $n > 0$. Since Collatz is true, there is a k such that $C^k(n) = 1$. Now let $\mathfrak{M}$ be any interpretation. If $\mathfrak{M} \models CA_1$, then $\mathfrak{M} \models R(\underline{n})$ by Lemma 3. Since this holds for any $\mathfrak{M}$, $(CA_1 \supset R(\underline{n}))$ is valid.

In the only-if direction, assume that $(CA_1 \supset R(\underline{n}))$ is valid for every $n > 0$. Then since $\mathfrak{G}_1 \models CA_1$, it follows that $\mathfrak{G}_1 \models R(\underline{n})$ for every $n > 0$. Hence $n \in R^{\mathfrak{G}_1}$ for every $n > 0$, and so by virtue of how $R^{\mathfrak{G}_1}$ is defined, Collatz is true. ∎

This theorem (and the fact that CA_1 consists of Horn clauses) shows that the Collatz conjecture reduces to whether or not the following non-arithmetic Prolog program terminates for all queries of the form `r(n)`:

```
t(o,o,o).
t(s(X),s(s(Y)),s(s(s(Z)))) :- t(X,Y,Z).
r(s(o)).
r(Y) :- t(X,Y,_), r(X).
r(s(Y)) :- t(_,Y,Z), r(s(s(Z))).
```

4 Second Variation

Although the theory CA_1 did not have arithmetic built-in, it did include axioms for the predicate T characterizing the doubling and tripling of numbers. At best, this is a weak form of arithmetic. (It is not at all clear that even addition can be defined in terms of T. I believe that it cannot, but I do not have a proof.) But it is still arithmetic. In this section, we consider a variation of the puzzle even further removed from numbers.

Definition 8 *By $\mathcal{L}_2$ we mean the logical language with four constant symbols, o, a, b, and c, one binary function symbol s, and one binary predicate symbol U.*

As will become clear below, the intended interpretation of this language is in terms of strings over the alphabet $\Sigma = \{\mathtt{A}, \mathtt{B}, \mathtt{C}\}$. But before getting to this, it is useful consider the following binary relations over these strings:

Definition 9 *Let* $PR = \{(\mathtt{A}, \mathtt{BC}), (\mathtt{B}, \mathtt{A}), (\mathtt{C}, \mathtt{AAA})\}$.

Definition 10 *Let* $\rightarrow\ = \{(uv\alpha, \alpha\beta) \mid u, v \in \Sigma,\ \alpha, \beta \in \Sigma^*,\ (u, \beta) \in PR\}$.

Definition 11 *For any* $k \geq 0$, *let* $\xrightarrow{k}\ \subseteq [\Sigma^* \times \Sigma^*]$ *be defined inductively by:*

- $\xrightarrow{0} = \{(\alpha, \alpha) \mid \alpha \in \Sigma^*\}$;
- $\xrightarrow{k+1} = \{(\alpha, \beta) \mid$ *for some* γ, $(\alpha, \gamma) \in \rightarrow$ *and* $(\gamma, \beta) \in \xrightarrow{k}\}$.

These three definitions provide a compact formulation of what is called a 2-tag system (invented by Emil Post [6] as a formal model of computation), here restricted to the following production rules (via the set *PR*):

```
A => BC
B => A
C => AAA
```

This particular 2-tag system was studied by Liesbeth De Mol [2]. She showed that $\mathtt{A}^{2i} \xrightarrow{k} \mathtt{A}^{i}$ for $k = 2i$, and that $\mathtt{A}^{2i+1} \xrightarrow{k} \mathtt{A}^{3i+2}$ for $k = 2i + 2$. This then follows:

Theorem 3 (de Mol) *Collatz is true iff for every* $n > 0$, $\mathtt{A}^n \xrightarrow{k} \mathtt{A}$ *for some* $k \geq 0$.

This means that the Collatz conjecture can be formulated in terms of the termination of the following program (written here using Python string notation):

```
x = 'A'*n
while len(x) > 1 do
  if x[0]=='A' then x += 'BC'
  if x[0]=='B' then x += 'A'
  if x[0]=='C' then x += 'AAA'
  x = x[2:]
end while
```

We now turn to a logical statement of these ideas. Note that a straightforward encoding of the above would require axioms characterizing the operation of appending a string onto the end of another arbitrarily large string:

$A(o, x, x)$
$A(x, y, z) \supset A(s(u, x), y, s(u, z))$

(where o here means the empty list). This is of course just a disguised version of the axioms for addition:

$$P(o, x, x)$$
$$P(x, y, z) \supset P(s(x), y, s(z))$$

(where o now means 0). This is to be expected since strings of the form $\mathtt{A}^n$ represent numbers in unary notation where appending represents addition. However, in what follows, we characterize the string operations without a separate append/addition predicate (inspired by difference lists in Prolog [7]).

Definition 12 *For any $\alpha \in \Sigma^*$ and term t, let $\alpha{\cdot}t$ stand for the term of $\mathcal{L}_2$ consisting of applications of s according to the characters in α and ending with t.*

So, for example, $\mathtt{ACCB}{\cdot}x$ is an abbreviation for $s(a, s(c, s(c, s(b, x))))$.

Definition 13 *Let CA_2 be the conjunction of the following two first-order sentences of $\mathcal{L}_2$ (where variables x, y are universally quantified from the outside):*

1. $U(\mathtt{A}{\cdot}o, o)$
2. $U(x, y) \supset$
   $U(\mathtt{AA}{\cdot}x, \mathtt{BC}{\cdot}y) \land U(\mathtt{AB}{\cdot}x, \mathtt{BC}{\cdot}y) \land U(\mathtt{AC}{\cdot}x, \mathtt{BC}{\cdot}y) \land$
   $U(\mathtt{BA}{\cdot}x, \mathtt{A}{\cdot}y) \land U(\mathtt{BB}{\cdot}x, \mathtt{A}{\cdot}y) \land U(\mathtt{BC}{\cdot}x, \mathtt{A}{\cdot}y) \land$
   $U(\mathtt{CA}{\cdot}x, \mathtt{AAA}{\cdot}y) \land U(\mathtt{CB}{\cdot}x, \mathtt{AAA}{\cdot}y) \land U(\mathtt{CC}{\cdot}x, \mathtt{AAA}{\cdot}y)$

Lemma 4 *Let $\mathfrak{M}$ be an interpretation such that $\mathfrak{M} \models CA_2$. Then for every $k \geq 0$ and $\alpha \in \Sigma^*$, if $\alpha \xrightarrow{k} \mathtt{A}$, then $\mathfrak{M} \models \exists z U(\alpha{\cdot}z, z)$.*

Proof: The proof is by induction on k.

Base case: If $\alpha \xrightarrow{0} \mathtt{A}$, then $\alpha = \mathtt{A}$ and the result holds by virtue of $CA_2^{(1)}$.

Induction step: Assume that for any α, if $\alpha \xrightarrow{k} \mathtt{A}$ then $\mathfrak{M} \models \exists z.\ U(\alpha{\cdot}z, z)$. Now suppose $\alpha \xrightarrow{k+1} \mathtt{A}$. Therefore, there is a $\lambda \in \Sigma^*$ such that $\alpha \to \lambda$ and $\lambda \xrightarrow{k} \mathtt{A}$. So by the induction hypothesis, $\mathfrak{M} \models \exists z.\ U(\lambda{\cdot}z, z)$. Since $\alpha \to \lambda$, there is $u, v \in \Sigma$ and $\beta \in \Sigma^*$ such that $\alpha = uv\alpha'$ and $\lambda = \alpha'\beta$, where $(u,\beta) \in PR$.

Given PR, there are three cases for u. If $u = \mathtt{A}$, then $\alpha = \mathtt{A}v\alpha'$ and $\beta = \mathtt{BC}$, and thus $\mathfrak{M} \models \exists z.\ U(\alpha'\mathtt{BC}{\cdot}z, z)$. Then by $CA_2^{(2)}$ (where we let x be $\alpha' v\mathtt{B}{\cdot}z$ and y be z), $\mathfrak{M} \models \exists z.\ U(\mathtt{A}v\alpha'\mathtt{BC}{\cdot}z, \mathtt{BC}{\cdot}z)$. Therefore (letting z' be $\mathtt{BC}{\cdot}z$), $\mathfrak{M} \models \exists z'.\ U(\alpha{\cdot}z', z')$. The other two cases of PR are similar. ∎

We now turn to our intended interpretation of the symbols in $\mathcal{L}_2$. We will use the following binary relation on strings:

Definition 14 *Let $\mathbb{Z} = \{(u_1 v_1 \ldots u_n v_n \mathtt{A}, \beta_1 \ldots \beta_n) \mid n \geq 0,\ u_i, v_i \in \Sigma,\ (u_i,\beta_i) \in PR\}$.*

Note that if $(uv\alpha, \beta\gamma) \in \mathbb{Z}$ where $(u, \beta) \in PR$, then $(\alpha, \gamma) \in \mathbb{Z}$. We also have the following:

Lemma 5 *If* $(\alpha\gamma, \gamma) \in \mathbb{Z}$ *where* $|\alpha| > 1$*, then for some* $k > 0$*,* $\alpha \xrightarrow{k} \mathtt{A}$.

Proof: The proof is by induction on $|\alpha\gamma|$.

Base case: When $|\alpha\gamma| = 0$, the lemma trivially holds since $|\alpha| \not> 1$.

Induction: Assume that $|\alpha\gamma| = m$ and that the lemma holds for any α', γ' such that $|\alpha'\gamma'| < m$. Now assume that $(\alpha\gamma, \gamma) \in \mathbb{Z}$ where $|\alpha| > 1$, so that $\alpha = uv\alpha'$ for some $u, v \in \Sigma$. Therefore, $\gamma = \beta\gamma'$ where $(u, \beta) \in PR$. There are three cases for u:

- $u = \mathtt{A}$. In this case, $\beta = \mathtt{BC}$. So $(\mathtt{A}v\alpha'\mathtt{BC}\gamma', \mathtt{BC}\gamma') \in \mathbb{Z}$, and therefore $(\alpha'\mathtt{BC}\gamma', \gamma') \in \mathbb{Z}$. Then $|\alpha'\mathtt{BC}| > 1$ and $|\alpha'\mathtt{BC}\gamma'| < m$. So by induction, there is a k such that $\alpha'\mathtt{BC} \xrightarrow{k} \mathtt{A}$. Since, $\alpha \to \alpha'\mathtt{BC}$, it follows that $\alpha \xrightarrow{k+1} \mathtt{A}$.
- $u = \mathtt{B}$. In this case, $\beta = \mathtt{A}$. There are two sub cases. If $\alpha' = \epsilon$, then since $\mathtt{B}v \to \mathtt{A}$, $\alpha \xrightarrow{1} \mathtt{A}$. Otherwise, we have that $(\mathtt{B}v\alpha'\mathtt{A}\gamma', \mathtt{A}\gamma') \in \mathbb{Z}$, and therefore $(\alpha'\mathtt{A}\gamma', \gamma') \in \mathbb{Z}$. Since $\alpha' \neq \epsilon$, $|\alpha'\mathtt{A}| > 1$ and $|\alpha'\mathtt{A}\gamma'| < m$. So by induction, there is a k such that $\alpha'\mathtt{A} \xrightarrow{k} \mathtt{A}$. Since, $\alpha \to \alpha'\mathtt{A}$, it follows that $\alpha \xrightarrow{k+1} \mathtt{A}$.
- $u = \mathtt{C}$. In this case, $\beta = \mathtt{AAA}$, and the rest is like the case $u = \mathtt{A}$.

This completes the proof ∎

Corollary 2 *For any* $n > 0$*, if* $(\mathtt{A}^n\gamma, \gamma) \in \mathbb{Z}$*, then for some* $k \geq 0$*,* $\mathtt{A}^n \xrightarrow{k} \mathtt{A}$.

Proof: For $n = 1$, we have that $\mathtt{A} \xrightarrow{0} \mathtt{A}$. For $n > 1$, Lemma 5 applies. ∎

Here is the intended interpretation of the symbols in $\mathcal{L}_2$:

Definition 15 *Let* $\mathfrak{G}_2$ *be the interpretation of* $\mathcal{L}_2$ *defined as follows:*

- *the domain of* $\mathfrak{G}_2$ *is* Σ^**, where* $\Sigma = \{\mathtt{A}, \mathtt{B}, \mathtt{C}\}$
- $o^{\mathfrak{G}_2} =$ *the empty string;*
- $a^{\mathfrak{G}_2} =$ *the string* A*;*
- $b^{\mathfrak{G}_2} =$ *the string* B*;*
- $c^{\mathfrak{G}_2} =$ *the string* C*;*
- $s^{\mathfrak{G}_2} = \{(u, \alpha, u\alpha) \mid u \in \Sigma, \alpha \in \Sigma^*\} \cup \{(u, \alpha, \alpha) \mid u \notin \Sigma, \alpha \in \Sigma^*\}$
- $U^{\mathfrak{G}_2} = \mathbb{Z}$.

This interpretation clearly satisfies CA_2 and leads to the main result of this section:

Theorem 4 *Collatz is true iff for every $n > 0$, the formula $(CA_2 \supset \exists x.\, U(\mathtt{A}^n \cdot x, x))$ is a valid sentence of first-order logic.*

Proof: In the if direction, suppose Collatz is true and that $n > 0$. By Theorem 3, there is a k such that $\mathtt{A}^n \xrightarrow{k} A$. Now let $\mathfrak{M}$ be any interpretation. If $\mathfrak{M} \models CA_2$, then $\mathfrak{M} \models \exists x.\, U(\mathtt{A}^n \cdot x, x)$ by Lemma 4. Thus $(CA_2 \supset \exists x.\, U(\mathtt{A}^n \cdot x, x))$ is valid.

In the only-if direction, assume that $(CA_2 \supset \exists x.\, U(\mathtt{A}^n \cdot x, x))$ is valid for every $n > 0$. Then since $\mathfrak{G}_2 \models CA_2$ it follows that $\mathfrak{G}_2 \models \exists x.\, U(\mathtt{A}^n \cdot x, x)$ for every $n > 0$. So for every $n > 0$, there is a $\gamma \in \Sigma^*$, such that $(\mathtt{A}^n\gamma, \gamma) \in U^{\mathfrak{G}_2}$. Then, by Corollary 2, there is a $k \geq 0$ such that $\mathtt{A}^n \xrightarrow{k} \mathtt{A}$. Therefore by Theorem 3, Collatz is true. ∎

It then follows that the Collatz conjecture reduces to whether or not the following Prolog program terminates for all queries of the form `u([a,...,a|X],X)`:

```
u([a],[]).
u([a,_|R],[b,c|T]) :- u(R,T).
u([b,_|R],[a|T]) :- u(R,T).
u([c,_|R],[a,a,a|T]) :- u(R,T).
```

5 Third Variation

In this third and final variation, we return to the first variation in terms of zero and successor, but using the insights of the second variation to avoid separate axioms for the doubling and tripling of numbers.

Definition 16 *By $\mathcal{L}_3$ we mean the logical language with one constant symbol o, one unary function symbol s, and one ternary predicate symbol Q.*

Definition 17 *Let CA_3 be the conjunction of the following four first-order sentences of $\mathcal{L}_3$ (where x, y, z are universally quantified from the outside):*

1. $Q(s(o), o, o)$
2. $Q(x, o, o) \supset Q(o, x, z)$
3. $Q(s(s(x)), o, o) \supset Q(s(o), y, x)$
4. $Q(x, s(y), s(s(s(z)))) \supset Q(s(s(x)), y, z)$

Again we will use $\underline{n}$ to stand for the ground term of $\mathcal{L}_3$ consisting of n applications of s to o. We have the following:

Lemma 6 *Let $\mathfrak{M}$ be an interpretation such that $\mathfrak{M} \models CA_3$. Then for $n, m \geq 0$, $\mathfrak{M} \models Q(\underline{m}, \underline{n}, \underline{3n}) \supset Q(\underline{2n+m}, o, o)$*

Proof: Apply axiom $CA_3^{(4)}$ n times. ∎

Lemma 7 *Let $\mathfrak{M}$ be an interpretation such that $\mathfrak{M} \models CA_3$. Then for any $n \geq 0$, $\mathfrak{M} \models Q(\underline{n}, o, o) \supset Q(\underline{2n}, o, o)$*

Proof: Suppose $\mathfrak{M} \models CA_3$ and $\mathfrak{M} \models Q(\underline{n}, o, o)$. Then by $CA_3^{(2)}$, $\mathfrak{M} \models Q(o, \underline{n}, \underline{3n})$. Then by Lemma 6, $\mathfrak{M} \models Q(\underline{2n}, o, o)$. ∎

Lemma 8 *Let $\mathfrak{M}$ be an interpretation such that $\mathfrak{M} \models CA_3$. Then for any $n \geq 0$, $\mathfrak{M} \models Q(\underline{3n+2}, o, o) \supset Q(\underline{2n+1}, o, o)$*

Proof: Suppose $\mathfrak{M} \models CA_3$ and $\mathfrak{M} \models Q(\underline{3n+2}, o, o)$. By $CA_3^{(3)}$, $\mathfrak{M} \models Q(\underline{1}, \underline{n}, \underline{3n})$. Then by Lemma 6, $\mathfrak{M} \models Q(\underline{2n+1}, o, o)$. ∎

Lemma 9 *Let $\mathfrak{M}$ be an interpretation such that $\mathfrak{M} \models CA_3$. Then for every $k \geq 0$ and $n > 0$, if $C^k(n) = 1$ then $\mathfrak{M} \models Q(\underline{n}, o, o)$.*

Proof: By induction on k.

Base case: If $k = 0$ and $C^k(n) = 1$ then $n = 1$ and $\mathfrak{M} \models Q(\underline{n}, o, o)$ by $CA_3^{(1)}$.

Induction step: Suppose $k \geq 0$ and if $C^k(n) = 1$ then $\mathfrak{M} \models Q(\underline{n}, o, o)$. Now suppose $n > 0$ and $C^{k+1}(n) = 1$. This means there is a $u > 0$ such that $C(n) = u$ and $C^k(u) = 1$. By the induction hypothesis, $\mathfrak{M} \models Q(\underline{u}, o, o)$. There are two cases for n: if n is even, where $n = 2u$, then $\mathfrak{M} \models Q(\underline{n}, o, o)$ by Lemma 7; if n is odd, say $n = 2v + 1$, where $u = 3v + 2$, then $\mathfrak{M} \models Q(\underline{n}, o, o)$ by Lemma 8. ∎

We now turn to the intended interpretation of $\mathcal{L}_3$. We need the following sets:

Definition 18 *Let $\mathbb{C} = \{n \mid$ for some $k \geq 0$, $C^k(n) = 1\}$. and $\mathbb{Q} = \mathbb{Q}_1 \cup \mathbb{Q}_2 \cup \mathbb{Q}_3$ where*

$\mathbb{Q}_1 = \{(n, 0, 0) \mid n \in \mathbb{C}\}$,
$\mathbb{Q}_2 = \{(2i, n - i, k) \mid k \geq 0,\ n \in \mathbb{C},\ i \leq n\}$,
$\mathbb{Q}_3 = \{(2i + 1, k, n - 3i) \mid k \geq 0,\ n + 2 \in \mathbb{C},\ 3i \leq n\}$.

Lemma 10 *For any $n > 0$, if $(n, 0, 0) \in \mathbb{Q}$ then $n \in \mathbb{C}$.*

Proof: Recall that by definition of C, if $i \in \mathbb{C}$ then $2i \in \mathbb{C}$, and if $3i + 2 \in \mathbb{C}$ then $2i + 1 \in \mathbb{C}$. Now assume $(x, 0, 0) \in \mathbb{Q}$. There are three cases: If $(x, 0, 0) \in \mathbb{Q}_1$, then $x \in \mathbb{C}$. If $(x, 0, 0) \in \mathbb{Q}_2$, then $x = 2i$ where $i = n$ and $n \in \mathbb{C}$, and so $i \in \mathbb{C}$, in which case $x = 2i \in \mathbb{C}$ also. Finally, if $(x, 0, 0) \in \mathbb{Q}_3$, then $x = 2i + 1$ where $3i = n$ and $n + 2 \in \mathbb{C}$, and so $3i + 2 \in \mathbb{C}$, in which case $x = 2i + 1 \in \mathbb{C}$ also. ∎

Definition 19 *Let $\mathfrak{G}_3$ be the interpretation of $\mathcal{L}_3$ defined as follows:*

- *the domain of $\mathfrak{G}_3$ is $\mathbb{N}$;*
- $o^{\mathfrak{G}_3} = 0$*;*
- $s^{\mathfrak{G}_3} = \{(n, n+1) \mid n \geq 0\}$*;*
- $Q^{\mathfrak{G}_3} = \mathbb{Q}$.

Lemma 11 $\mathfrak{G}_3 \models CA_3$.

Proof: For $CA_3^{(1)}$, because $1 \in \mathbb{C}$, $(1, 0, 0) \in \mathbb{Q}_1$.

For $CA_3^{(2)}$, suppose that $(x, 0, 0) \in \mathbb{Q}$. Then by Lemma 10, $x \in \mathbb{C}$. Then letting $i = 0$, $(0, x, k) \in \mathbb{Q}_2$ for every k.

For $CA_3^{(3)}$, suppose that $(x+2, 0, 0) \in \mathbb{Q}$. Then by Lemma 10, $(x+2) \in \mathbb{C}$. Then letting $i = 0$, $(1, k, x) \in \mathbb{Q}_3$ for every k.

Finally, for $CA_3^{(4)}$, suppose that $(x, y+1, z+3) \in \mathbb{Q}$. There are two cases for x:

- if x is even, then $(2i, n-i, k) \in \mathbb{Q}_2$, where $x = 2i$, $y+1 = n-i$, $z+3 = k$ and $n \in \mathbb{C}$. So $i+1 \leq n$ and $k \geq 3$. Letting $i' = i+1$ and $k' = k-3$, we have that $(2i', n-i', k') \in \mathbb{Q}_2$, that is, $(x+2, y, z) \in \mathbb{Q}_2$.
- if x is odd, then $(2i+1, k, n-3i) \in \mathbb{Q}_3$, where $x = 2i+1$, $y+1 = k$, $z+3 = n-3i$ and $n+2 \in \mathbb{C}$. So $3i+3 \leq n$ and $k \geq 1$. Letting $i' = i+1$ and $k' = k-1$, we have that $(2i'+1, k', n-3i') \in \mathbb{Q}_3$, that is, $(x+2, y, z) \in \mathbb{Q}_3$.

So either way, $(x+2, y, z) \in \mathbb{Q}$. ∎

Here then is the main result of this section:

Theorem 5 *Collatz is true iff for every $n > 0$, the formula $(CA_3 \supset Q(\underline{n}, o, o))$ is a valid sentence of first-order logic.*

Proof: In the if direction, suppose that $n > 0$. Since Collatz is true, there is a k such that $C^k(n) = 1$. Now suppose $\mathfrak{M} \models CA_3$. Then $\mathfrak{M} \models Q(\underline{n}, o, o)$ by Lemma 9. Since this holds for any interpretation, $(CA_3 \supset Q(\underline{n}, o, o))$ is valid.

In the only-if direction, assume that $(CA_3 \supset Q(\underline{n}, o, o))$ is valid for every $n > 0$. Then by Lemma 11, $\mathfrak{G}_3 \models CA_3$ and so $\mathfrak{G}_3 \models Q(\underline{n}, o, o)$ for every $n > 0$. Hence $(n, 0, 0) \in \mathbb{Q}$ for every $n > 0$. So by Lemma 10, $n \in \mathbb{C}$ for every $n > 0$, and hence Collatz is true. ∎

As a consequence of this theorem, the Collatz conjecture reduces to the claim that the following program terminates for any initial value of the variable `n`:

```
while n > 1 do
  i := j := 0
  while n > 1 do
    n := n - 2
    i := i + 1
    j := j + 3
  end while
  n := if n < 1 then i else j + 2
end while
```

Here, we see the Collatz conjecture at its purest. Multiplication and division are certainly not needed, but neither is full addition! The conjecture can be stated quite concisely in terms of adding or subtracting the constants 1, 2, and 3.

6 Coda

In each of the three variations above, we ended up with theorems of this form:

> The following formula $\phi[x]$ with free variable x has the property that the Collatz conjecture is true iff for every $n > 0$, the sentence $\phi[\underline{n}]$ is logically valid in first-order logic.

It would have been nicer, of course, if we had been able to reduce the conjecture not to an infinite set of validity questions but to the validity of a single sentence.

We can, of course, do this in second-order logic. It is perhaps worth noting in conclusion that the full power of second-order logic is not needed to do this. For example, a small extension to first-order logic will do the trick:

Definition 20 *Single-value transitive-closure logic is exactly like first-order logic except that for any unary function symbol f, we allow f^* to be used as a binary predicate. Formulas are interpreted as usual, except that the predicate f^* is always interpreted as the reflexive transitive closure of the function f.*

Now returning to the language $\mathcal{L}_1$, for example, we have the following:

Theorem 6 *Collatz is true iff $(CA_1 \supset \forall x.\ s^*(o, x) \supset R(s(x)))$ is a valid sentence of first-order single-value transitive-closure logic.*

Proof: In the if direction, assume that Collatz is true. Let $\mathfrak{M}$ be any interpretation with domain E such that $\mathfrak{M} \models CA_1$. Let $\sigma = s^{\mathfrak{M}}$ and $e_0 = o^{\mathfrak{M}}$. Now suppose that e is any element of E such that $\mathfrak{M}, \mu_e^x \models s^*(o, x)$. Then (e_0, e) is in the reflexive transitive closure of σ, so there is an $n \geq 0$ such that $e = \sigma^n(e_0)$. Since Collatz is true, there is a $k \geq 0$ such that $C^k(n+1) = 1$. By Lemma 3, $\mathfrak{M} \models R(s(\underline{n}))$ and so $\mathfrak{M}, \mu_e^x \models R(s(x))$. Thus, for any $e \in E$, if $\mathfrak{M}, \mu_e^x \models s^*(o, x)$, then $\mathfrak{M}, \mu_e^x \models R(s(x))$. So $\mathfrak{M} \models \forall x. s^*(o, x) \supset R(s(x))$. Since this holds for any $\mathfrak{M}$ such that $\mathfrak{M} \models CA_1$, it follows that $(CA_1 \supset \forall x.\ s^*(o, x) \supset R(s(x)))$ is valid.

In the only-if direction, assume that $(CA_1 \supset \forall x.\ s^*(o, x) \supset R(s(x)))$ is valid. Since $\mathfrak{G}_1 \models CA_1$, it follows that $\mathfrak{G}_1 \models \forall x.\ s^*(o, x) \supset R(s(x))$. Since for every $n \geq 0$, $\mathfrak{G}_1 \models s^*(o, \underline{n})$, it follows that for every $n \geq 0$, $\mathfrak{G}_1 \models R(s(\underline{n}))$. Hence $n \in R^G$ for every $n > 0$, and so by definition of R^G, Collatz is true. ∎

Comparing this to Corollary 1, we see that the real difficulty in reducing Collatz to a single sentence of pure first-order logic is not the arithmetic at all; it is the problem of restricting quantification to just the successors of zero.

7 Conclusion

In the world of mathematics, nothing is more basic than arithmetic over the natural numbers. To teach mathematics to children, we start with $1 + 1 = 2$ and go from there. But while arithmetic is the starting point in pedagogical terms, it is not the starting point in logical terms. More basic than a logic with arithmetic is one without. In this paper, we looked at the Collatz conjecture with this in mind, showing three ways that it could be reformulated in logic without assuming all the machinery of arithmetic.

Reducing Collatz to logic in this way raised some questions that we have not answered. Perhaps the most fundamental is the one touched on in the coda, namely, whether or not we can write a single sentence ψ of first-order logic such that the Collatz conjecture is true iff ψ is logically valid. Of course such a sentence ψ must *exist*: if the conjecture is true, any tautology will do; if it is false, the negation of any tautology will do. More interesting is whether we can exhibit such a ψ without having to solve the puzzle.

References

[1] A. Bes, A survey on arithmetic definability, *Belgian Mathematical Society*, 1–54, 2002.

[2] L. de Mol, Tag systems and Collatz-like functions, *Theoretical Computer Science*, **390**(1), 92–101, 2008.

[3] H. B. Enderton. *A Mathematical Introduction to Logic*, Academic Press, New York, 2001.

[4] J. C. Lagarias, The $3x + 1$ problem and its generalizations, *The American Mathematical Monthly*, **92**(1), 3–23, 1985.

[5] J. C. Lagarias (ed.), *The Ultimate Challenge: The* $3x + 1$ *Problem*, American Mathematical Society, Providence, RI, 2010.

[6] E. Post, Formal reductions of the combinatorial decision problem, *American Journal of Mathematics*, **65**(2), 197-215, 1943.

[7] Leon Sterling and Ehud Shapiro, The Art of Prolog, The MIT Press, Cambridge, Massachusetts, 1994.