

## Basic Prerequisite Mathematics

*This handout contains a review of basic mathematical notation and concepts used throughout the course. Please read it carefully and make sure that you know the material well enough to use it from previous courses. The material will not be covered in lectures or tutorials.*

### Set Theory

#### Common Sets

- $\mathbb{N}$ : the natural numbers, or non-negative integers ( $= \{0, 1, 2, \dots\}$ )  
NOTE: 0 is a natural number!
- $\mathbb{Z}$ : the integers ( $= \{\dots, -2, -1, 0, 1, 2, \dots\}$ )
- $\mathbb{Z}^+$ : the positive integers ( $= \{1, 2, 3, \dots\}$ )
- $\mathbb{Z}^-$ : the negative integers ( $= \{-1, -2, -3, \dots\}$ )
- $\mathbb{Q}$ : the rational numbers (and  $\mathbb{Q}^+$  the positive rationals,  $\mathbb{Q}^-$  the negative rationals)
- $\mathbb{R}$ : the real numbers (and  $\mathbb{R}^+$  the positive reals,  $\mathbb{R}^-$  the negative reals)

#### Notation

For any sets  $A$  and  $B$ , we will use the following standard notation.

- $x \in A$ : “ $x$  is an element of  $A$ ” or “ $A$  contains  $x$ ”
- $A \subseteq B$ : “ $A$  is a subset of  $B$ ” or “ $A$  is included in  $B$ ”
- $A = B$ : “ $A$  equals  $B$ ” (note that  $A = B$  if and only if  $A \subseteq B$  and  $B \subseteq A$ )
- $A \cup B$ : “ $A$  union  $B$ ”
- $A \cap B$ : “ $A$  intersection  $B$ ”
- $A \setminus B$  or  $A - B$ : “ $A$  minus  $B$ ” (*set difference*)
- $|A|$ : “cardinality of  $A$ ” (the number of elements of  $A$ )
- $\emptyset$  or  $\{\}$ : “the empty set”
- $\mathcal{P}(A)$ : “powerset of  $A$ ” (the set of all subsets of  $A$ )

For example, if  $A = \{a, 34, \Delta\}$ , then

$$\mathcal{P}(A) = \{\{\}, \{a\}, \{34\}, \{\Delta\}, \{a, 34\}, \{a, \Delta\}, \{34, \Delta\}, \{a, 34, \Delta\}\}.$$

- $\{x \mid P(x)\}$  (where  $P(x)$  is some property of  $x$ ): “the set of elements  $x$  for which  $P(x)$  is true”  
For example,  $\{x \in \mathbb{Z} \mid x^2 < 9\}$  represents the set of integers  $x$  for which  $x^2$  is less than nine, *i.e.*, it is equal to  $\{-2, -1, 0, 1, 2\}$  and also to  $\{x \in \mathbb{Z} \mid -7 \leq 3x \leq 6.4\}$ .

## Relations

- a *tuple* or vector  $(x_1, \dots, x_n)$  is an ordered collection of  $n$  objects.  
When  $n = 2$ , the tuple is called an *ordered pair*.  
When  $n = 1$ , the parentheses are usually omitted.
- for any sets  $A_1, \dots, A_n$ , the *Cartesian product* of  $A_1, \dots, A_n$  is a set of tuples defined by

$$A_1 \times \cdots \times A_n = \{(x_1, \dots, x_n) \mid \text{for every } i \text{ from } 1 \text{ to } n, x_i \in A_i\}.$$

Example:  $\{1, a\} \times \mathbb{N} = \{(1, 0), (a, 0), (1, 1), (a, 1), (1, 2), (a, 2), (1, 3), \dots\}$ .

Properties: when the sets  $A_i$  are all finite,  $|A_1 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$ .

- for any set  $A$  and natural number  $n$ , the notation  $A^n$  is used to denote the Cartesian product of  $A$  with itself:

$$\underbrace{A \times A \times \cdots \times A}_{n \text{ times}}.$$

When  $n = 1$ ,  $A^n$  is the set  $A$ .

When  $n = 0$ ,  $A^n$  is considered to be the set  $\{()\}$ , which has one element, the 0-tuple,  $()$ .

- for any sets  $A_1, \dots, A_n$ , a *relation over*  $A_1, \dots, A_n$  is any subset of  $A_1 \times \cdots \times A_n$ .  
When  $n = 2$ , the relation is called a *binary relation*.

Example:  $R = \{(x, y) \mid x < y, x \in \mathbb{N}, y \in \mathbb{N}\}$  is a binary relation over  $\mathbb{N}$  and  $\mathbb{N}$ . This  $R$  is called “the less-than relation over  $\mathbb{N}$ ”. Note that this  $R$  is also a relation over  $\mathbb{R}$  and  $\mathbb{Z}$ .

- for any sets  $A$  and  $B$ , a *function  $f$  from  $A$  to  $B$* , written

$$f : A \rightarrow B$$

is any subset of  $A \times B$  where for each  $x \in A$  there is a unique  $y \in B$  such that  $(x, y) \in f$ .

We write  $f(x)$  to refer to the  $y$  associated with  $x$  by  $f$  (called the value of  $f$  at  $x$ );

when  $A$  is the Cartesian product  $A_1 \times \cdots \times A_n$ , we write  $f(x_1, \dots, x_n)$ .

Example:  $\min(x, y)$ : “minimum of  $x$  and  $y$ ” (the smallest of  $x$  or  $y$ ).

This function satisfies  $\min(x, y) \leq x$ , and  $\min(x, y) \leq y$ .

## Basic Number Theory

For any two natural numbers  $a$  and  $b$ , we say that  $a$  *divides*  $b$  if there exists a natural number  $c$  such that  $b = ac$ . In such a case, we say that  $a$  is a *divisor* of  $b$  (e.g., 3 is a divisor of 12 but 3 is not a divisor of 16). Note that any natural number is a divisor of 0 and 1 is a divisor of any natural number.

A natural number  $p$  is *prime* if it has exactly two positive divisors (e.g., 2 is prime since its positive divisors are 1 and 2 but 1 is **not** prime since it only has one positive divisor: 1). There are an infinite number of prime numbers and any integer greater than one can be expressed in a unique way as a finite product of prime numbers (e.g.,  $8 = 2^3$ ,  $77 = 7 \times 11$ ,  $3 = 3$ ).