CSC 330F 2005, Assignment 2, St. George Campus
Due: Nov. 8th, 2:10 PM

On the cover page of your assignment, you must list everyone with whom you discussed this assignment, and which problems you discussed with each person. You must also write **and sign** *the following statement: "I have read and understood the policy on collaboration on homework assignments stated in the Course Information handout." Without these, your homework will not be marked.*

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1. Let F be the sentence $\exists y_1 \exists y_2 \forall x (R(x, y_1) \vee R(x, y_2))$. Let G be the sentence $\exists y_1 \forall x \exists y_2 (R(x, y_1) \wedge R(x, y_2))$. [10]
   (a) Prove using structures that F logically entails G or prove that it does not.
   (b) Prove using structures that G logically entails F or prove that it does not.

2. Consider the following word problem: [25]

   Mary is a member of Club Northtown or Club Northridge.
   All the members of Club Northtown play racquetball or squash.
   The members of Club Northridge who are athletic play tennis.

   In this question, we consider logical entailments of these facts.

   (a) Represent the three given facts as sentences in a first-order logical language with four constant symbols, m (Mary), r (racquetball), s (squash), and t (tennis), three unary predicate symbols, NT (is a member of Club Northtown), NR (is a member of Club Northridge), A (is athletic) and one binary predicate symbol P (plays). Call the set of sentences Σ.
   (b) Write a sentence F in this language saying the following: if Mary is athletic, then she plays something. Prove using structures that Σ entails F.
   (c) Write a sentence G in this language saying that Mary plays something. Prove using structures that Σ does not entail G.
   (d) Does Σ entail $\neg G$? Explain your answer informally.

3. This question concerns formalizing simple properties of *sets* as axioms in a first-order logical language with equality. Consider the following facts: [25]
   - No set is an element of itself.
   - A set x is a subset of a set y iff every element of x is an element of y.
   - For every set x and y, there is a unique set z called the union of x and y.
   - Something is an element of the union of sets x and y iff
     it is an element of x or an element of y.

The language we will use has no constant symbols, but two binary predicate symbols, $Ele(x, y)$ (x is an element of y) and $Sub(x, y)$ (x is a subset of y), and a ternary predicate symbol $Uni(x, y, z)$ (z is the union of x and y).

(a) Represent the above facts as a set Σ of sentences in this language. Note that there is no predicate to assert that something is a set. You may simply assume that everything (of interest) is a set.

(b) Prove using structures that Σ entails that for any sets x and y, x is a subset of the union of x and y.

(c) Prove that Σ does not entail that the union of sets x and y is equal to the union of y and x.

(d) Does Σ entail that for every set x there is a set y such that the union of x and y is a subset of x? Explain your answer informally.

(e) Does Σ entail that there is a set y such that for every set x the union of x and y is a subset of x? Explain your answer informally.

4. Let $\mathcal{L}$ be the relational language for students, courses, *etc.* and $\mathcal{R}$ be the relational database over this language, as presented in class. [10]

(a) Write the predicate completion of the *Prereq* predicate over $\mathcal{R}$. Call it F.

(b) Let Σ be the (infinite) set of all sentences of $\mathcal{L}$ that are true in $\mathcal{R}$. Does Σ logically entail F? Explain your answer informally.

5. Consider the following rule of inference: [10]

$$\text{IF } \Sigma \models \forall x(F_1 \wedge \cdots \wedge F_n \supset G) \text{ THEN } \Sigma \cup \{F_1[x/c], \ldots, F_n[x/c]\} \models G[x/c].$$

Prove using structures that this rule is sound. You may use without proof the following fact:

> **Theorem**: Let F be any formula with at most x free, let c be any constant, and let $\mathcal{S}$ be any structure for a language containing F and c. Then $F[x/c]$ is true in $\mathcal{S}$ iff F is true in $\mathcal{S}$ for x as $c^{\mathcal{S}}$.

6. Prove the entailment of Question 2b using a derivation. In this question, you may use the rule of inference from Question 5 as an additional rule. *Hint:* With the additional rule, there is a derivation with 13 nodes. [20]

Bonus question: This question is a continuation of Question 3.

Write a sentence T in the language of the question that asserts the existence of singleton sets, that is, for any x, the set whose only element is x. Prove that $\Sigma \cup \{T\}$ is not finitely satisfiable (that is, the sentences cannot all be true in a structure with a finite domain). *Hint:* In a finite domain, consider u, the object interpreted as the union of all the elements in the domain.

Do not attempt any bonus work until the regular part of your assignment is complete.